* We know the reason for $J \gtrless 0$
* We know how to formulate a bosonic theory for quantum magnets.

1. Introduction

Last time we have seen that at half-filling, the fermionic Hubbard model maps to the Heisenberg-model

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \vec{S}_j$$

if $t \ll U$. Here $\vec{S}_i$ are spin-operators with

$$[S_i^\alpha, S_j^\beta] = i\hbar \delta_{ij} \varepsilon_{\alpha\beta\gamma} S_i^\gamma .$$

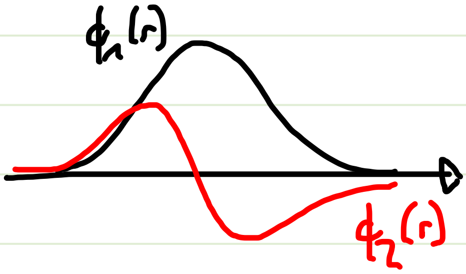
The fact that the different spin components do not commute makes this Hamiltonian intrinsically quantum mechanical! Another way to see this is to write

$$\frac{1}{J} H_{ij} = \vec{S}_i \vec{S}_j = S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z = \underbrace{\frac{1}{2}\left[S_i^+ S_j^- + S_i^- S_j^+\right]}_{\text{order is } x\text{-}y} + \underbrace{S_i^z S_j^z}_{\text{order in } z}$$

We try to see what consequences we can expect from such non-commuting terms. Before embarking on this program, we want to understand what determines the sign of J.

2. Ferro- vs. antiferromagnetism

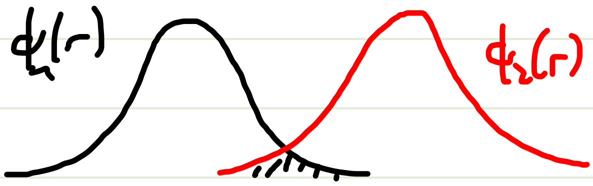
a.) Fermions in overlapping orbitals:



$\Rightarrow$ "Hund's" rule: spin polarize $\Rightarrow$ different orbitals $\Rightarrow$ smaller overlap $\Rightarrow$ less interaction energy.

$\Rightarrow$ $J < 0$: Ferromagnet

b) Fermions in weakly overlapping orbitals:



$\Rightarrow$ energy reduction by tunneling between orbitals

$\Rightarrow$ Ferromagnetic ordering would prevent hopping due to Pauli blocking

$\Rightarrow$ $J >$: Antiferromagnet

c.) Bosons in weakly overlapping orbitals:

⚠

kinetic energy is "Bose enhanced" $b_i|n_0\rangle = \sqrt{n_0}\,|n_0 - 1\rangle$

$\Rightarrow$ same-spin leads to larger kinetic energy $\Rightarrow$

$J < 0$: Ferromagnet

3. Schwinger and Holstein-Primakoff bosons

We need an efficient way to deal with quantum fluctuations in the spin-Hamiltonian. We apply the following program

(i) Find a good classical state

(ii) Deal with fluctuations around this classical state.

For the second point we want to make use of our knowledge of interacting bosonic systems: spin-operators at different sites commute & if we don't deal with "too much fluctuations" the fact that bosonic and spin Hilbertspaces are different might not matter too much.

Let us introduce constraint bosons, i.e., Holstein-Primakoff bosons

$$S^+ = \sqrt{2S - n_b}\, b$$

$$S^- = b^+ \sqrt{2S - n_b} \qquad n_b = b^+ b \qquad n_b \leq 2S$$

$$S^z = S - n_b$$

Using $[b, b^\dagger] = 1$, it is easy to show that $[S^\alpha, S^\beta] = i\varepsilon_{\alpha\beta\gamma} S^\gamma$

What do we profit from such a description? In a symmetry broken phase we assume that all spin classically point down, i.e., $S^z = n_b = 0$ in the classical ground state.

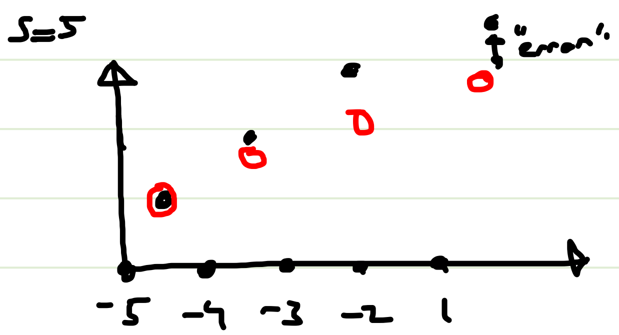
(In case of an anti-ferromagnet on a bi-partite lattice we first rotate all spins on one sub-lattice).

In a next step, we make an expansion of the square root:

$$\sqrt{2S - n_b} \approx \sqrt{2S}\left[1 - \frac{n_b}{4S} - \frac{n_b^2}{32S^2} + \dots\right]$$

A few comments:

(i) It is apparently an expansion in $\frac{1}{S}$. This reflects that if we have a large spin, deviations from the fully polarized state are almost bosonic.



(ii) lowest order expansion leads to a quadratic bosonic problem.

In the exercises we will see how this program can be applied to the Heisenberg anti-ferromagnet. The most important result will be:

$$\Delta m = \langle \vec{S} \rangle_{classical} - \langle \vec{S} \rangle_{q.m.},$$

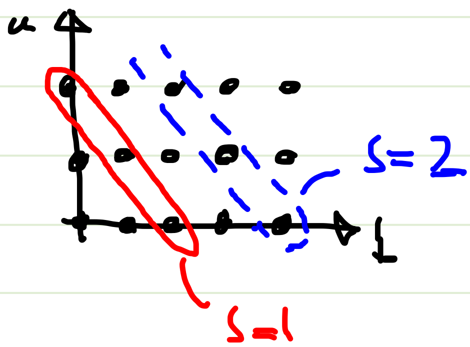
i.e. the fluctuation induced reduction of the (staggered) magnetic moment. This exactly corresponds to the quantum depletion of a bosonic condensate.

While Holstein-Primakoff bosons are good to describe a symmetry-broken state ($n_b \ll 0$), it is not a-priori clear how to find this symmetry broken state in general. For a symmetric starting point, it is better to use Schwinger bosons:

$$S^+ = a^\dagger b$$

$$S^- = b^\dagger a \qquad\qquad n_a + n_b = 2S$$

$$S^z = \frac{1}{2}(a^\dagger a - b^\dagger b)$$



We can now find the optimal g.s. by making unitary transformation in the a-b space. Then we proceed by

$$n_{\tilde{a}} + n_{\tilde{b}} = 2S \Rightarrow n_{\tilde{a}} = 2S - n_{\tilde{b}} \Rightarrow \tilde{a}^{+} = \sqrt{2S - n_{\tilde{b}}}$$

We can now make the step to H.P.:

$$b \longleftrightarrow b \; ; \; a \longleftrightarrow \sqrt{2S - n_b}$$

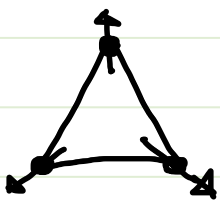
4. Interesting examples and applications

a.) Spin-Liquids

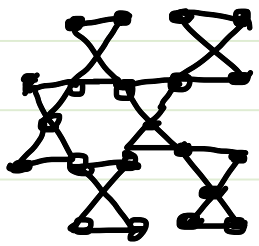
Spin-systems that avoid ordering even at zero temperatures
How can this happen?

Quantum-fluctuation reduce ordering → if order is weak to begin with: SL

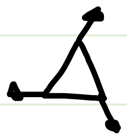
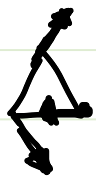
no classical order because of frustration → order-by-disorder does not fix the problem



b.) Frustrated spin-systems: an example



$$H = J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)$$

⇒ on every triangle:  or 

⇒ one triangle does not fix everything

⇒ $\langle e^{i\varphi} \rangle = 0$ <u>but</u> $\langle e^{3i\varphi} \rangle \neq 0$

⇒ $\frac{1}{3}$ - vortices

c) strong correlations in bosonic systems: truncate Hilbert-space, interpret it as spin ⇒ use H.P.