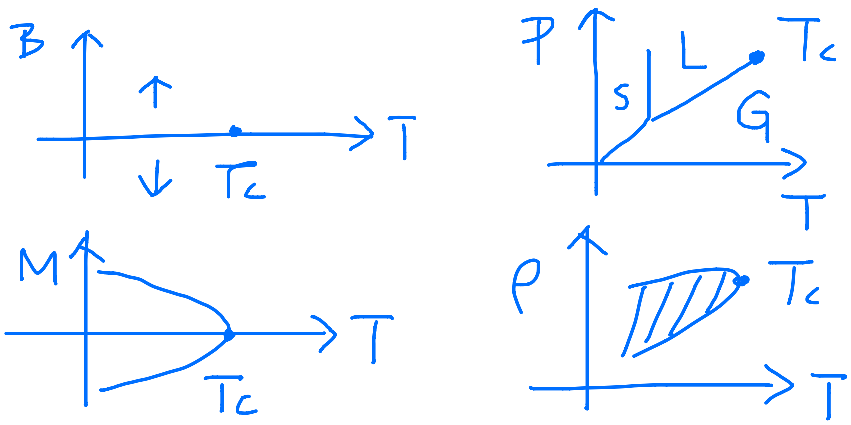
Quantum phase transition: scale invariance and universality

1. Classical vs. quantum phase transition?
2. Mean field approach?
3. Experiment evidence? Constraints?
4. Scale invariance in quantum critical gas

Examples of classical phase transitions:
Liquid-gas transition 1st order vs. 2nd order transition
2nd order liquid-gas transition is the same as Ising transition!!



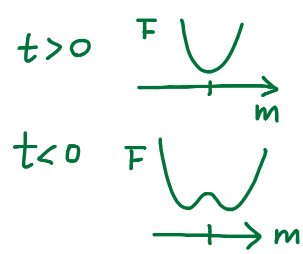
HW1: Critical exponents in classical Landau mean field theory
Define t=(T-Tc)/Tc, as critical exponent Λ(X,t)>0 of an observable X near t=0 is given by the asymptotic behavior of

$$X \sim |t|^{\Lambda} \quad \text{or} \quad \Lambda = \lim_{t \to 0} \frac{\ln|X|}{\ln|t|}.$$

Three critical exponents can be defined this way: α=Λ(C,t), β=Λ(m,t) and γ=Λ(χ,t), where C is the heat capacity, m is the magnetization and χ is the magnetic susceptibility. $(\chi = \partial m / \partial B |_T)$

Landau assumed free energy F(T,B)=E-TS of a ferromagnet at field B can be expanded near the critical point as

$$F = F_0 + A t m^2 + \lambda m^4 + mB, \text{ where } \lambda > 0.$$



Calculate α, β and γ.

Mean field theory applied to bosons in optical lattices

$$H = -t\sum_{\langle ij\rangle}\left(a_i^{\dagger}a_j + a_j^{\dagger}a_i\right) + \frac{U}{2}\sum_i n_i(n_i-1) - \mu\sum_i n_i$$

Mean field approximation: $|\varphi\rangle = \prod_i |\phi\rangle_i$, mean-field H_{MF}

$$H_{MF} = -zt\sum_i\left(c\,a_i + c^* a_i^{\dagger}\right) + \frac{U}{2}\sum_i n_i(n_i-1) - \mu\sum_i n_i, \quad c \equiv \langle a\rangle, \quad c^* \equiv \langle a^{\dagger}\rangle$$

Assume $|\phi\rangle = |m\rangle$, $m = 0, 1, 2\cdots$

perturbation gives $|m'\rangle = |m\rangle + \sum_{n\neq m} |n\rangle \frac{\langle n|H_{MF}|m\rangle}{E_m - E_n}$

$$E^{(0)} = \frac{U}{2}m(m-1) - \mu m$$

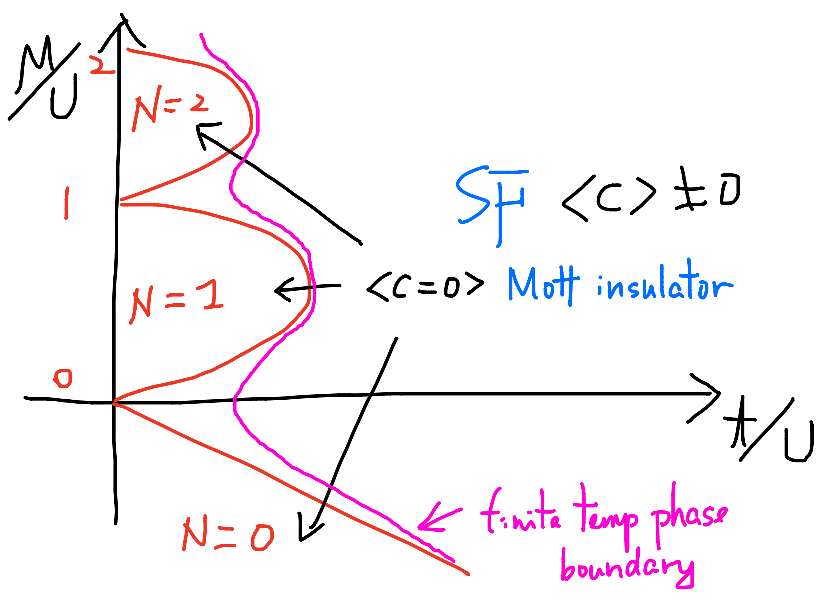
$$E^{(1)} = \langle m'|H - H_{MF}|m'\rangle$$

$$= \mathcal{E}_0 + \mathcal{E}_2|c|^2 + \mathcal{E}_4|c|^4 + \cdots$$

Phase transition occurs when $\mathcal{E}_2$ flips sign & $\mathcal{E}_4 > 0$

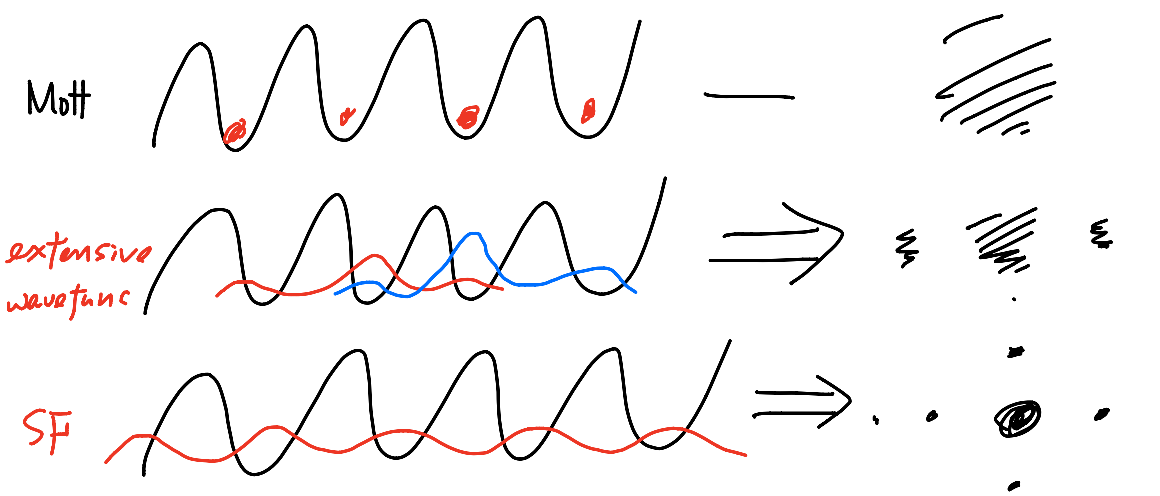
$\mathcal{E}_2 = 0$ gives $\dfrac{zt}{U} = \dfrac{(m-\mu/U)(\mu/U - m + 1)}{\mu/U + 1}$

c is the order parameter



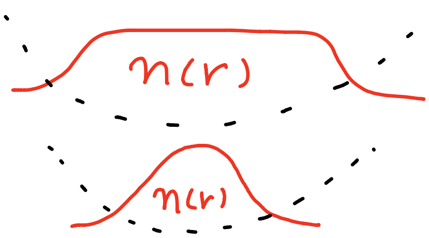
Experimental evidence

1. Diffraction pattern from the lattices



2. Compressibility

Mott insulator $\qquad \kappa_T = \partial n / \partial \mu |_T = 0$



Superfluid $\qquad = 1/g$

criticality: $\kappa \sim T^{-1-d/z+2/\nu z}$

3. Fluctuations

$$\text{Mott} \quad \Delta n^2 = 0, \quad \text{SF} \quad \Delta n^2 \sim \frac{1}{g}.$$

$$\sum_j \delta n_i \delta n_j = kT\kappa$$

4. Equation of state

$$n - n_c = T^{\frac{D}{z}+1-\frac{1}{z\nu}} F\left(\frac{\mu-\mu_c}{T^{1/z\nu}}\right)$$

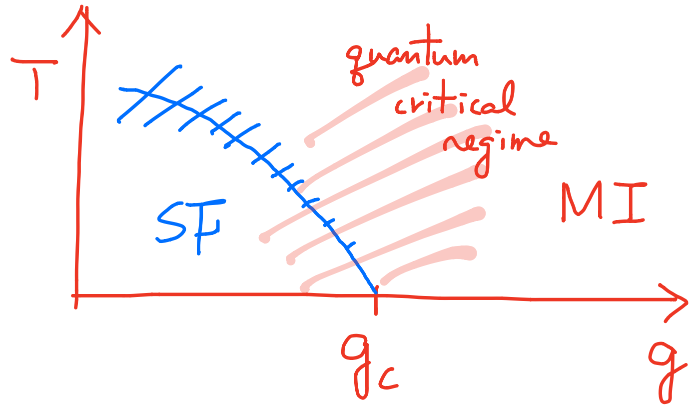
critical chemical potential

correlation exponent.

dynamic exponent.

How do we test these predictions?

Quantum criticality



Theoretical construct:

partition function $Z = \mathrm{Tr}\, e^{-H/KT}$ $\quad \tau \sim \hbar/KT$

$$\approx \int D\vec{b}\, e^{-\int_0^{\hbar/KT} \mathcal{L}\, d\tau}$$

$$\mathcal{L} = \Sigma\, {db}/{d\tau}\, b - H$$

Expansion with low energy and momentum and small order parameter

$$\mathcal{L} = \underbrace{K_1 \varphi^* \frac{\partial \varphi}{\partial \tau} + K_2 \left|\frac{\partial \varphi}{\partial \tau}\right|^2}_{\substack{\text{imaginary time} \\ \tau \sim 1/KT}} + \underbrace{K_3 |\nabla \varphi|^2}_{\substack{\text{spatial} \\ (\text{K.E.})}} + \underbrace{r|\varphi|^2 + u|\varphi|^4}_{\substack{\text{mean field} \\ r = g - g_c}}$$

Case 1 $K_1 \neq 0$. K_2 not important

$$g = g_c \qquad \begin{aligned} &x \to \lambda x \\ &\varphi \to \lambda^{-D/2} \varphi \\ &\tau \to \lambda^2 \tau. \Rightarrow T \to \lambda^{-1} T \Rightarrow \varepsilon \sim p^{z=2} \end{aligned}$$

$g \neq g_c$ $\quad r \to \lambda^{-2} r. \Rightarrow$ correlation length $\xi \sim (g-g_c)^{-\nu}$

$\nu = 1/2$

Case 2: $K_1 = 0$

$$\begin{aligned} &x \to \lambda x \\ &\tau \to \lambda \tau \\ &T \to \lambda^{-1} T \Rightarrow \varepsilon \sim p^{z=1} \\ &r \to \lambda^{-2} r \Rightarrow \xi \sim (g-g_c)^{-\cancel{1/2}} \end{aligned}$$

-0.67 in 2D

anomymous dimension needs RG.

HW2 In recent spin-orbit coupled exps, the single particle dispersion can be controled as $E \sim \varepsilon p^2 + p^4$. When ε drops below 0, it supports 2 minima. Extend the BH Lagrangian to include the new E and analyze the critical behavior near $\varepsilon = 0$.

Hint: consider 1D for simplicity. The Lagrangian density is

$$\mathcal{L} = k_1 \varphi^* \partial_z \varphi + \varepsilon |\partial_z \varphi|^2 + |\partial_z^2 \varphi|^2 + k_3 |\nabla \varphi|^2 + r|\varphi|^2 + u|\varphi|^4$$

Does it change the critical behavior when $k_1 \neq 0$?