Chapter 2

Waves in solids

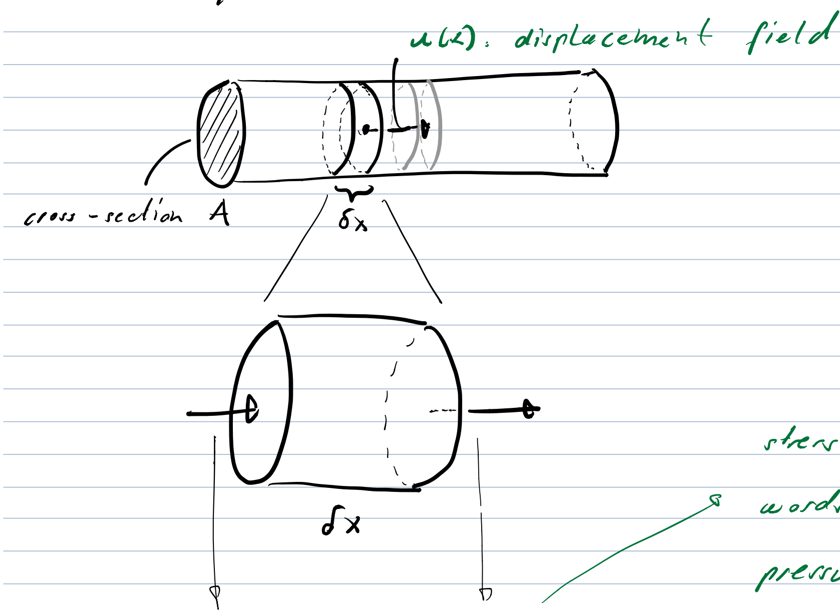
Learning goals

- You know the wave equation.
- You know that concepts and differences between the group and phase velocity.

2. Waves in solids

To familiarise ourselves with waves and their propagation, we study a simple example of longitudinal waves in thin rods.

Let us consider the forces acting on a small section of the rod:



stress field, or in other words, force per area: pressure

$$F = -A\sigma(x) + A\sigma(x+\delta x)$$
$$= -A\left(\sigma(x) + \sigma(x) + \delta x\, \sigma'(x)\right) = A\,\delta x\, \sigma'(x)$$

We know that Newton's equations of motion govern the behaviour of mechanical systems. Hence

$$ma = F \Rightarrow \text{for our cross-section}$$

$ma = A\,\delta x\, \rho\, \ddot{u}(x,t)$ and therefore we find

$$A\,\delta x\,\rho\,\ddot{u}(x,t) = A\,\delta x\,\sigma'(x,t) \qquad (1)$$

In order to close this equation, we need a relation between the stress $\sigma(x,t)$ and the displacement $u(x,t)$. As all segments of a finite rod can take up some of the applied force, the change in length of an elastic object depends on its size.

$$\sigma = E\,\frac{\Delta L}{L} \qquad \text{or} \qquad \sigma(x,t) = E\,u'(x,t)$$

with a proportionality fact E called Young's modulus. Inserting this relation into Equation (1) we obtain

$$\boxed{\ddot{u}(x,t) = \frac{E}{\rho}\,u''(x,t)} \qquad (2)$$

This is our sought wave equation. Let us analyze a few properties of this equation.

2.1 Travelling wave solutions

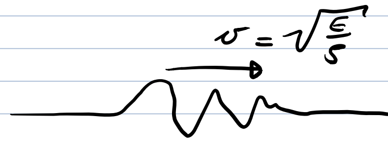
Any (twice differentiable) function

$$u(x - vt) \qquad \text{with } v =$$

is a solution of (2). To see this let us insert this ansatz into the wave equation

$$\frac{\rho}{E}\,v^2\,u''(x-vt) = u''(x-vt)$$

$\Rightarrow$ if $v = \sqrt{\frac{E}{\rho}}$ $u(x - vt)$ is indeed a solution



We read off a few interesting properties from this solution

a.) Any travelling wave-form is preserved under the evolution of time.

b.) The stiffer the material (E larger), the faster the wave.

c.) The lighter the material (ρ smaller), the faster the wave.

While these observations are important and useful, we profit from another analysis in terms of "modes."

2.2 Eigenmodes of the wave equation

An eigenmode is a natural vibration of the system where all parts oscillate at the same frequency. It is useful to know these modes, as we can construct all solutions from a super-position of such eigenmodes.

For the wave equation these are simple travelling waves

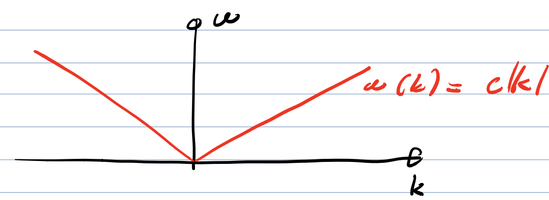
$$u(x,t) = e^{i\varphi(x,t)} = e^{i(kx - \omega t)}$$

Inserted into (2) we find

$$-c^2 k^2 e^{i(kx-\omega t)} = -\omega^2 e^{i(kx-\omega t)}$$

$$\Rightarrow \quad \boxed{\omega(k) = c|k| \quad \text{with} \quad c = \sqrt{\frac{E}{\rho}}}$$

This connection between the wave-number k and the angular frequency ω is called the dispersion relation.



Most of what we are going to do in this lecture is trying to manipulate $\omega(k)$ to achieve our design goals.

Let us therefore understand a few more properties of the dispersion relation.

2.3 The phase velocity

We have seen in the first lecture that the phase of a wave is a very important property. The point in space of constant phase φ_0 is moving in time

$$0 = \varphi_0(x,t) = kx - \omega(k)\cdot t \quad \Rightarrow \quad x_0 = \underbrace{\frac{\omega(k)}{k}} t$$

arbitrary choice

v_φ : phase velocity

where we defined the phase velocity v_φ.

The phase velocity will become particulary relevant for two and three-dimensional systems, where a non-isotropic $\omega(\vec{k})$ (i.e. $\omega(\vec{k}) \neq c|\vec{k}|$) can lead to a distortion of the wave front.

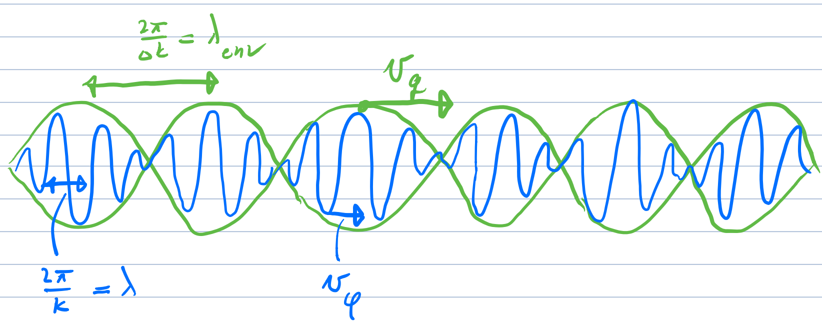
2.4 The group velocity

As we are often dealing with wave-packets [like the one depicted in the beginning $u(x-vt)$], the phase velocity is not the only important quantity.

Imagine a wave-packet made from modes that are close to a base frequency ω:

$$u(x,t) = e^{i(\omega+\frac{\Delta\omega}{2})t} e^{i(k+\frac{\Delta k}{2})x} + e^{i(\omega-\frac{\Delta\omega}{2})t} e^{i(k-\frac{\Delta k}{2})x}$$

$$= 2e^{ikx-i\omega t}\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$



The envelope $\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$ travels with the velocity $v_g = \frac{\Delta\omega}{\Delta k}$. If we take the limit of "$\Delta \to 0$" we find

$$v_g = \frac{\partial \omega(k)}{\partial k} \quad \text{group velocity.}$$

In the exercises we will see when $v_g = v_\varphi$ and what is the significance of the group velocity.

2.5 The wave number k

We have seen that $e^{ikx - \omega(k)t}$ are solutions to the wave equation (2). Note that the wave number k encodes the wavelength $\lambda = \frac{2\pi}{k}$. Moreover k controls how the phase of the wave changes when we move in space:

$$\arg[u(x,t_0)] = \arg[e^{ikx - i\omega(k)t}] = kx - \omega(k)t$$

$$\arg[u(x+\Delta x, t_0)] = k(x+\Delta x) - \omega(k)t$$

$$\Rightarrow \boxed{\Delta\varphi = k\,\Delta x}$$

And this is the only thing that changes in $u_k(x,t)$ when advancing by Δx. On solutions characterized by "k" we can think that translations act by multiplying by:

$$T_{\Delta x} = e^{i\Delta x \cdot k}$$

As we will see in the next chapter, this simple property might be lost if our medium is not translationally invariant. And this is what we typically do in a metamaterial design: We structure a material in a way that breaks continous translation-symmetry, e.g., by drilling holes.

2.6 A note on waves in solids

In the above example we study one simple type of elastic wave that occurs in solids. Of course there are many more types of waves like shear-waves, torsional waves, flexural waves, waves the only propagate on surfaces like Love or Rayleigh waves. For the purpose of this lecture, however, we focus much more on waves in discrete systems, as for our metamaterial concepts, they are more instructive.