Chapter 6

The continuum limit

Learning goals

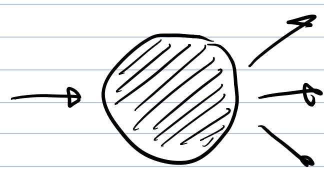
- You know how to define effective parameters for a discrete system.
- You know how to find the effective mass for a simple setup.
- You know how to determine the effective spring constant.

6. The continuum limit

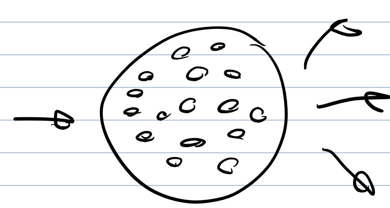
In chapters 3 & 4 we have seen how we can shape the propagation of waves by using periodic structures or by employing local resonances. In the last chapter, on the other hand, we have seen how one can obtain interesting wave phenomena in a simple wave equation, however with "unnatural" parameters $\varepsilon<0$ and $\rho<0$.

How to map a structured metamaterial to a simple wave equation with effective parameters is a highly non-trivial task. We can group approaches to do so into two classes. First, if we deal with a continous medium to which we add resonances of periodic arrays of scatterers, one typically does the following:

1.) Solve how a circular object with ε, ρ, ν scatters a wave.



2.) Solve the problem of inclusions in a circular metamaterial and calculate how it scatters an incoming wave. 3.) By comparing the two results, fix ε, ρ, ν.



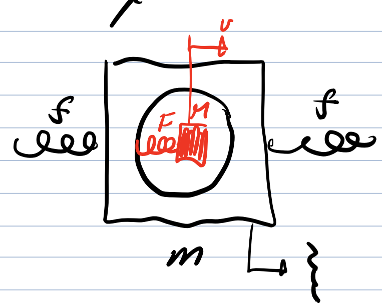
Here, we would like to learn how to map the discrete models with interesting band structures to effective "continuum" theories. In this case we introduce "natural" coordinates ξ_i and think of the blue body as rigid, despite

the fact that they may have an inner line. The effective mass m_{eff} and effective spring constant are then defined via

$$m_{\text{eff}} := \frac{F_i}{\ddot{\xi}_i} \quad \leftarrow \text{force on } i \qquad \text{and} \qquad f_{\text{eff}}^{ij} = \frac{F_{ij}}{\xi_i - \xi_j} \quad \leftarrow \text{force between } i \text{ and } j .$$

6.1 Mass in mass

We start by revisiting the mass-in-mass system.



Imagine that we would not see the interior mass M. So we should find an effective description for the "outer" mass only. Let us write

$$M \ddot{v} = F(\xi - v) \tag{1}$$

we assume harmonic motion

$$v = v_0 e^{i\omega t}$$
$$\xi = \xi_0 e^{i\omega t}$$

Inserted into (1) we get

$$-\omega^2 M v_0 = F(\xi_0 - v_0)$$

and therefore

$$v_0 = \frac{F\xi_0}{F - \omega^2 M}$$

by writing $\omega_M = \sqrt{\frac{F}{M}}$ we get

$$v_0 = \frac{\omega_M^2}{\omega_M^2 - \omega^2}\xi_0 \,. \qquad (2)$$

Let us now introduce the effective mass. We know that

$$\underbrace{\frac{dP}{dt}}_{\text{change in total momenta}} = \frac{\partial}{\partial t}\left(m\frac{\partial \xi}{\partial t} + M\frac{\partial v}{\partial t}\right) = \underbrace{F_{ext}}_{\text{applied force}} \qquad (3)$$

As we do not "know" about the mass M we would write

$$m_{eff}\frac{\partial^2 \xi}{\partial t^2} = P \,.$$

As (2) is independent of time we have

$$\frac{dv}{dt} = \frac{\omega_\mu^2}{\omega_\mu^2 - \omega^2} \frac{\partial \xi}{\partial t}.$$

Using this expression in (3) we find

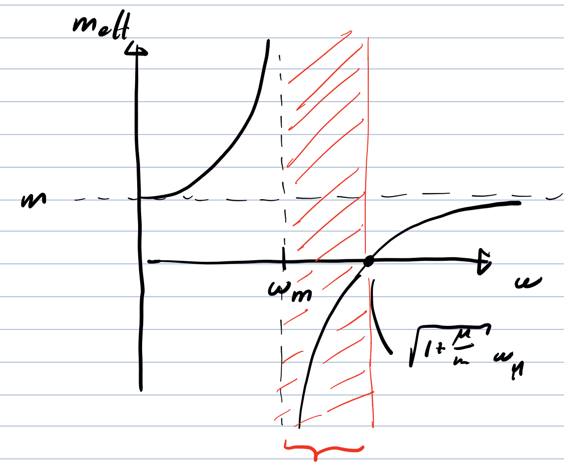
$$\frac{d}{dt}\left(m \frac{\partial \xi}{\partial t} + M \frac{\omega_\mu^2}{\omega_\mu^2 - \omega^2} \frac{\partial \xi}{\partial t} \right) = F_{ext}$$

$$\frac{\partial}{\partial t}\left[m \left(1 + \frac{M}{m} \frac{\omega_\mu^2}{\omega_\mu^2 - \omega} \right) \frac{\partial \xi}{\partial t} \right] = F_{ext}$$

$$m_{eff} \frac{\partial^2 \xi}{\partial t^2} = F_{ext}$$

where we defined

$$m_{eff} = m\left[1 + \frac{M}{m} \frac{\omega_\mu^2}{\omega_\mu^2 - \omega^2} \right]$$



negative effective mass!

- for $m_{eff} < 0$: You push and the mass moves in your direction
- m_{eff} is strongly frequency dependent!

- $\omega \to 0$ $\quad m_{eff} = m + M$

Let us now use this effective mass description to solve the chain of such local resonators:

$$m_{eff}(\omega)\,\frac{d^2\xi_i}{dt^2} = -f\left[2\xi_i - \xi_{i+1} - \xi_{i-1}\right].$$

As usual we use $\xi_i = e^{ikr_i}$ to obtain

$$-m_{eff}(\omega)\,\omega^2 = -f\left[2 - e^{-ika} - e^{ika}\right]$$

$$m_{eff}(\omega)\,\omega^2 = 2f\left[1 - \cos ka\right] = 4f\sin^2\frac{ka}{2}$$

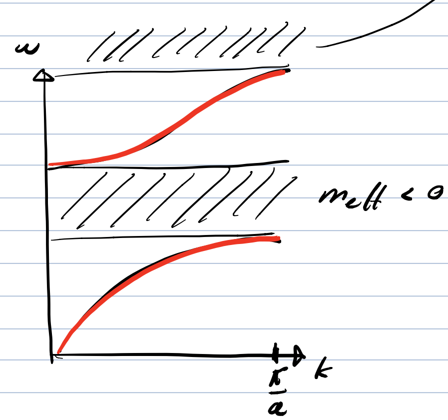
We solve for k:

$$k = \frac{2}{a}\arcsin\left[\frac{1}{2}\sqrt{\frac{m_{eff}(\omega)}{f}}\,\omega\right].$$

From this expression we see that k becomes imaginary if

1.) $m_{eff} < 0$ $\quad\Leftarrow$ Band gap

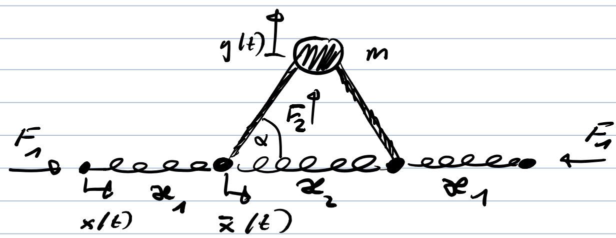
2.) $\frac{1}{2}\sqrt{\frac{m_{eff}}{f}}\,\omega > 0$ $\quad\Leftarrow$ above band edge



6.2 Effective spring

We have seen that a "hidden" mass can give rise to an effective negative mass. What about negative springs?

Let us consider the following system



The equations of motion are

$$m\ddot{y} = F_2$$

$$F_2 = 2(F_1 - 2\varkappa_2 \bar{x})\tan\alpha$$

$$F_1 = \varkappa_1(x - \bar{x})$$

$\Rightarrow \quad \bar{x} = x - \frac{F_1}{\varkappa_1} \quad \Rightarrow \quad f = 2\left(F_1 - 2\varkappa_2\left(x - \frac{F_1}{\varkappa_1}\right)\right)\tan\alpha$

and $\quad \bar{x} = \tan\alpha \, y \quad \Rightarrow \quad y = \frac{1}{\tan\alpha}\left(x - \frac{F_1}{\varkappa_1}\right)$

$$\Rightarrow \quad -m\omega^2\left(x - \frac{F_1}{\varkappa_1}\right) = 2\left(F_1 - 2\varkappa_2\left(x - \frac{F_1}{\varkappa_1}\right)\right)\tan^2\alpha$$

$$\Rightarrow \quad x = F_1 \frac{m\omega^2 - 2(\varkappa_1 + 2\varkappa_2)\tan^2\alpha}{\varkappa_1(m\omega^2 - 4\varkappa_2\tan^2\alpha)}$$

We can now define

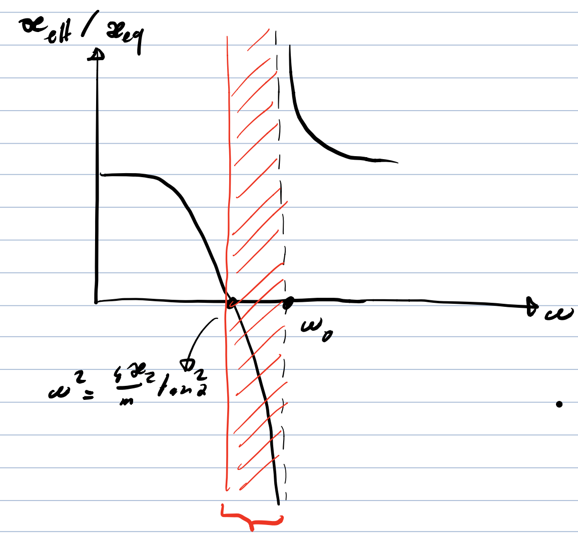
$x = \frac{F_1}{2k_{eff}}$ and write

$$k_{\text{eff}} = \frac{1}{2}\left[\frac{\left(\omega^2 - \frac{4\varkappa_2}{m}\tan^2\alpha\right)\varkappa_1}{\omega^2 - \omega_0^2}\right]$$

with $\omega_0^2 = \frac{2(\varkappa_1 + 2\varkappa_2)}{m}\tan^2\alpha$. Let us check for the sanity of this result by taking

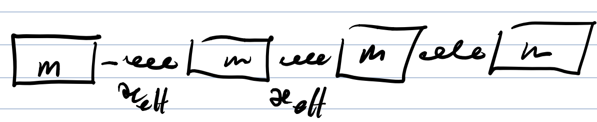
$$\lim_{\omega \to 0} \frac{1}{\varkappa_{\text{eff}}} = \frac{1}{\varkappa_2} + \frac{2}{\varkappa_1} = \frac{1}{\varkappa_{eq}},$$

which is what we should obtain! Analogous to the effective mass, can the effective spring constant be negative:



negative spring constant

Again bunching them together



$\Rightarrow$ The equations of motion read

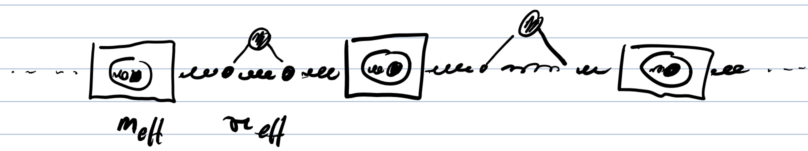
$$m\ddot{\xi}_i = -\varkappa_{eff}\left(2\xi_i - \xi_{i+1} - \xi_{i-1}\right)$$

$$\Rightarrow \quad k = \frac{2}{a}\arcsin\left(\frac{1}{2}\sqrt{\frac{m}{\varkappa_{eff}}}\,\omega\right)$$

And we have again a band gap where $\varkappa_{eff} < 0$!

6.4 Double negativity

Let us now combine the two ingredients, an effective mass and an effective spring:



$$\Rightarrow \quad m_{eff}\,\omega^2 = 4\varkappa_{eff}\sin^2\left(\frac{ka}{2}\right)$$

Task: Write m_{eff} and $\varkappa_{eff}$ in a way that transparently captures the negative sections. Then solve the above equation for ω and plot the solutions as a function of k and vary the tuning parameters. Convince yourself that indeed for $m_{eff}\cdot\varkappa_{eff} < 0$ you end up in a band gap and for $m_{eff} < 0$ & $\varkappa_{eff}$ you have $\frac{\partial\omega}{\partial k} < 0$.

References

1. Zhou, X., Liu, X. & Hu, G. "Elastic metamaterials with local resonances: an overview". *Theor. Appl. Mech. Lett.* **2,** 041001 (2012).
2. Lee, S. H. & Wright, O. B. "Origin of negative density and modulus in acoustic metamaterials". *Phys. Rev. B* **93,** 024302 (2016).