Chapter 7

Topological mechanical metamaterials

Learning goals

- You can sketch the history of topological bandstructures.
- You know the Su-Schrieffer-Heeger model.
- You know what a Chern insulator is.

7. Topological mechanical metamaterials

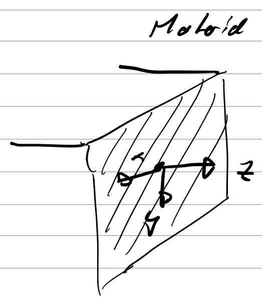
In this chapter we introduce a new approach to mechanical metamaterials which is inspired a a topic know in modern solid state physic: topological bandstructures. Before we present the main concepts we discuss the main application where topology might have a considerable impact in the near future: waveguiding.

7.1 Mechanical waveguides

The precise control of the flow of mechanical energy is an important task for applications ranging from energy harvesting or vibration isolation to the design of mechanical logic or signal processing. Let us introduce of few design principles for waveguides.

7.1.1. Surface acoustic waves

One way to control the flow of vibrations is simply to take a half-space filled with some elastic material:



The wave equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

permits solutions of the form

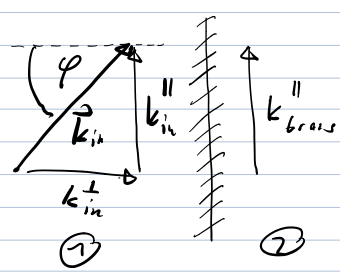
$$e^{ik_x x + ik_y y - z/\xi - i\omega t} \Rightarrow \boxed{-\frac{\omega^2}{c^2} = -k_x^2 - k_y^2 + \frac{1}{\xi}}$$

$\Rightarrow$ waves can be confined to the surface around $z=0$ but propagate in the x-y plane. To really match the boundary conditions, one needs to have either transverse (Love) or mixed (Rayleigh) polarizations.

What is important for our purposes is, that one can use surfaces as acoustic wave guides where the energy is falling off only with $\frac{1}{r}$ rather than $\frac{1}{r^2}$ for a propagation in 3 dimensions.

7.1.2 Total internal reflection

Even more spatial confinement is offered by using total internal reflection. Remember that for a boundary we had to match the parallel wave number



$$k_{in}^{\parallel} = k_{trans}^{\parallel}.$$

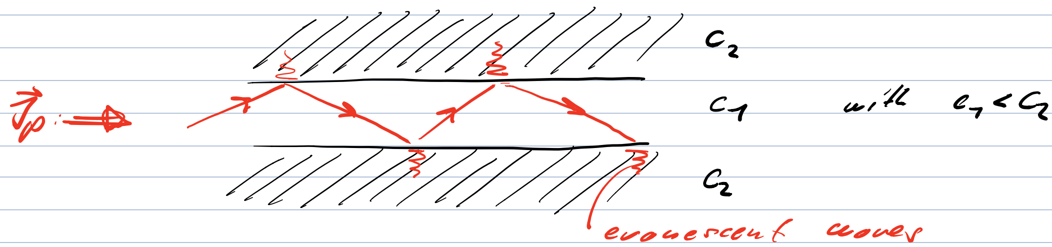
But we also have

$$k_{trans}^{\perp} = \pm\sqrt{\left(\frac{\omega}{c_2}\right)^2 - k_{trans}^{\parallel}}.$$

We see that if $c_2 > c_1$ and the angle of incident φ is too large, $k_{trans}^{\perp}$ will become imaginary. That means that above a certain critical angle

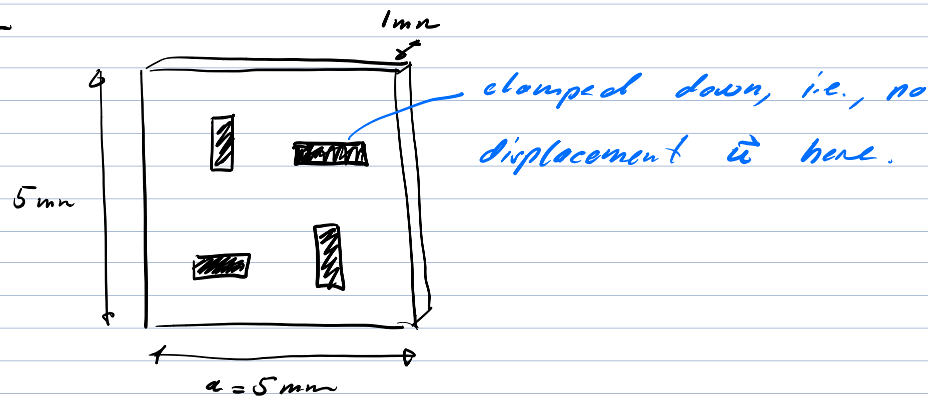
$$\varphi_{crit} = \arcsin\left(\frac{n_2}{n_1}\right)$$

the waves experience total internal reflection:

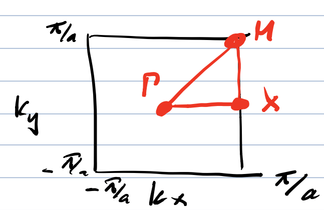


7.1.3 Band gap wave guides

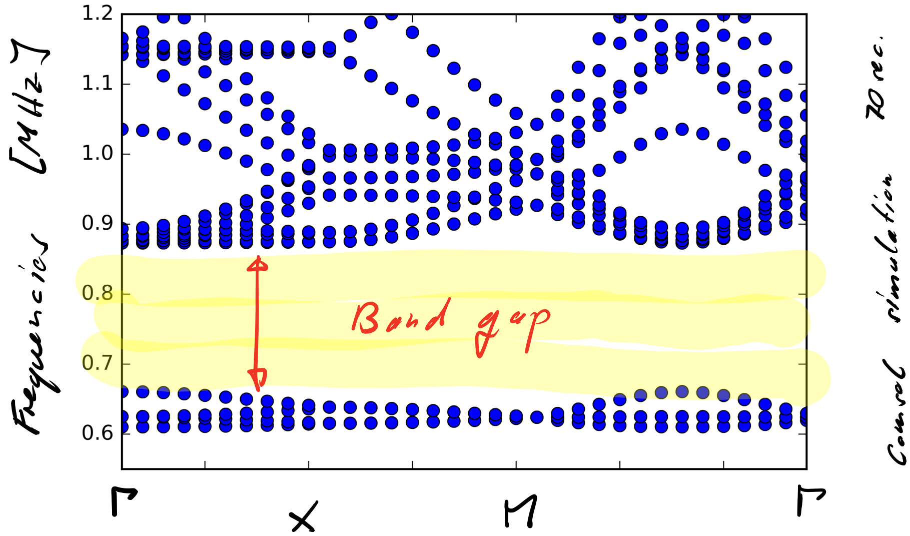
We know another way to get evanescent waves! Let us design a phononic crystal in a 2 dimensional sheet of Aluminum. If we take a unit cell of the form



and extend it in both directions, we get a Brillouin zone

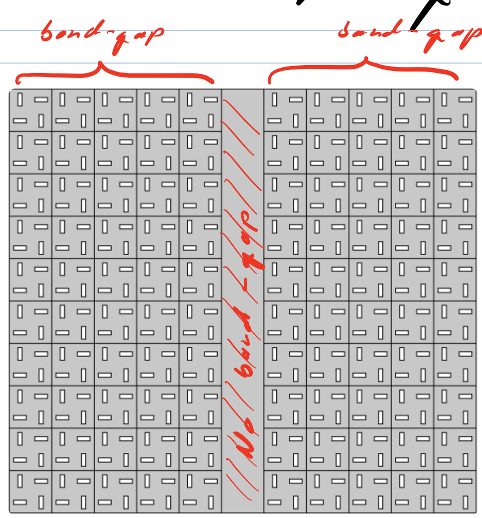


The spectrum along the high-symmetry line Γ-X-M-Γ looks like



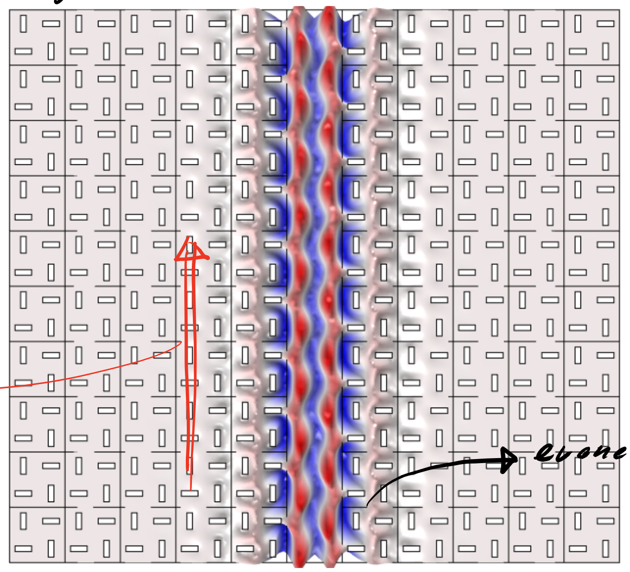
Do the frequencies make sense? $v_{AL} \sim 6000\,\frac{m}{s}$ and $a = 5\cdot 10^{-3}\,m \Rightarrow \nu_{edge} \approx \frac{v_{AL}}{a} \approx 10^{6}\,Hz$ ✓

From this plot we learn that from about 680 – 850 kHz, no waves can travel through our structure. This observation enables the following wave guide design.



We can expect, that for frequencies of 680 – 850 kHz, the middle strip without the clamped regions acts as a perfect wave-guide. The following mode analysis with COMSOL supports this expection:

Eigen mode at ~750 kHz



travelling waves.

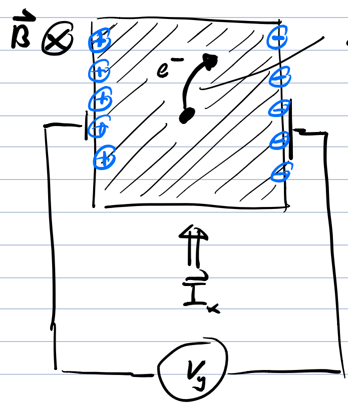
evanescent waves

shown is the out of plane displacement of the eigenmode at 750 kHz.

We have now seen a collection of possible wave-guiding structures. While some can be fairly small and maybe even reliable for straight segments, they all suffer from the same problem: Imperfections in the fabrication will always introduce disorder, which in turn can scatter the wave back to where it came from! For reliable waveguides we might profit from an entirely new design principle!

7.2 A short history on topology in condensed matter physics.

An early instance of topological effects relevant for transport physics was discovered in 1980 in the context of electron transport in a strong magnetic field.



Lorentz force.

For a two-dimensional electron system, a magnetic field exerts a force on moving charges which is perpendicular to their velocity

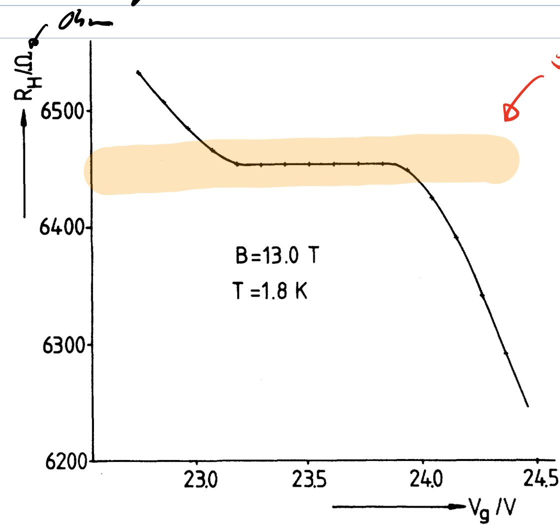
$$\vec{F}_L = -|e|\,\vec{v} \wedge \vec{B}.$$

This leads to a build-up of charges at the edges and introduces a voltage across the sample. This effect is quantified by the Hall conductance

$$\sigma_H = \frac{I_x}{V_y}. \qquad \left(\rho_H = \frac{1}{\sigma_H} \right)$$

Clearly σ_H will depend on the parameters of the system such as density n_e, the magnetic field strength B, etc.

In his famous measurement in 1980, Klaus von Klitzing discovered the following behavior:



flat at 6453 Ω

PRL, 45 494 (1980)

changes the electron density

It turns out that

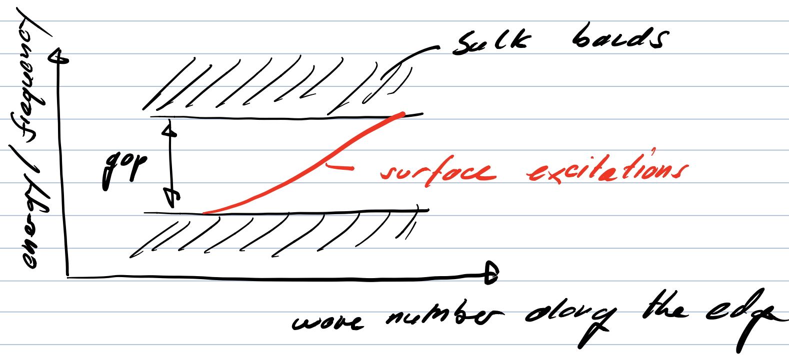
$$6453\,\Omega = \frac{h}{4e^2},$$

and the flatness of the plateau allows to determine the von Klitzing constant $\frac{h}{e^2}$ with a precision of 10^{-9}!

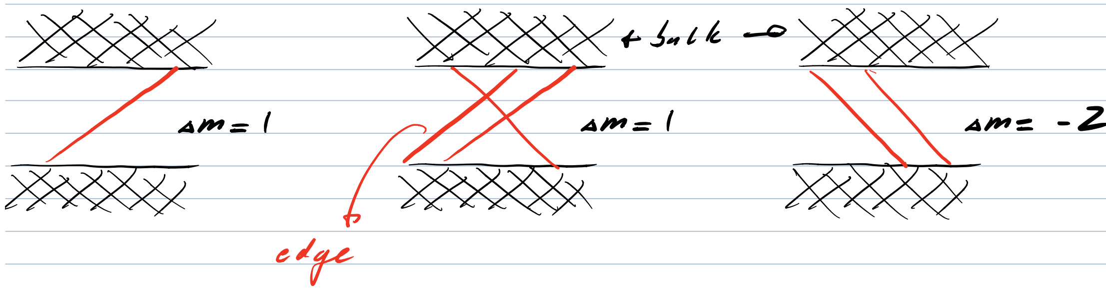
The Nobel prize for physics of 2016 was given to Thouless, Kosterlitz and Haldane for uncovering the topological origin of this precision. Thouless and co-workers could show that σ_H is given by the Chern number, a topological quantity that depends on the system as a whole, takes only integer values and can therefore only by changed in steps:

$$\sigma_H = m \frac{e^2}{h} \qquad m \in \mathbb{Z}. \text{ (Chern number)}$$

Underlying their calculation is the existence of a spectral gap. If we now think of a finite sample with an edge, it is easy to show that $m = 0$ in vacuum. If $m \neq 0$ inside the sample, it has to change across the boundary from $m \neq 0 \to 0$. This is only possible if the spectral gap is closing at the edge!



In fact one can assign a Chern number to each band. The bulk-edge correspondence tells us that the difference in Chern number above and below a gap $\Delta m = m_{above} - m_{below}$ dictates the signed number of surface channels. "Signed" means we count each channel times the sign of its group velocity.



The fact that the Chern number $m=1$ (left picture) predicts a uni-directional channel is responsible for the high precision of the flatness of the plateau in σ_H. If we add disorder to the system, electrons travelling on the red branch cannot scatter and turn around!

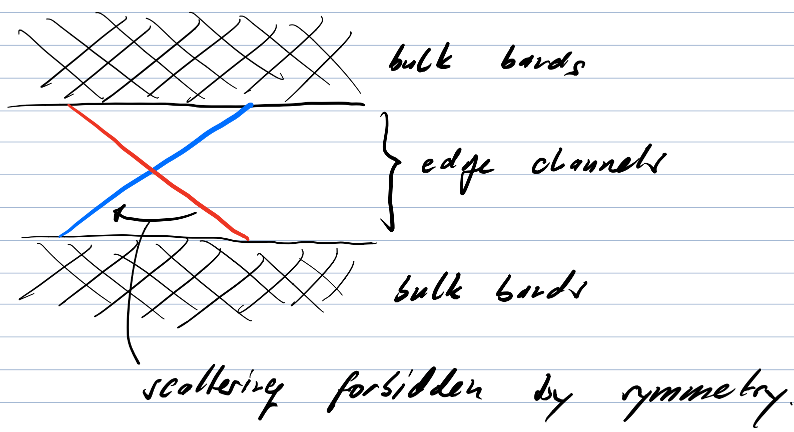
The electrons travel in a strict one way street!

Such stable features that make the transport of electrons extremely robust is what we seek for vibrations in metamaterials!

For a wide use both in electronics as well as in phononics the strong magnetic field needed for the quantum Hall effect is a bit of a problem. In electronics it is annoying, in phononics almost impossible.

A discovery by Kane & Mele and a subsequent experiment by a group led by Zhang and Molenkamp in 2005 - 2007 came as a cure to this problem.

Kane and Mele realized that under the right circumstances a topological band structure can exist also for time-reversal symmetric systems, i.e., in the absence of a magnetic field. In what is called a topological insulator, counter-propagating edge channels exist, but scattering between them is forbidden by a symmetry of the system.



For the case of topological insulators the topological quantum number is not an integer $m \in \mathbb{Z}$ but

$$m \in \mathbb{Z}_2 = \{0, 1\}$$

and indicates is there is an odd (even) number of pairs of counter-propagating surfaces states $m=1$ $(m=0)$.

In the following we are going to discuss a few simple lattice model that illustrate how the bulk of a system