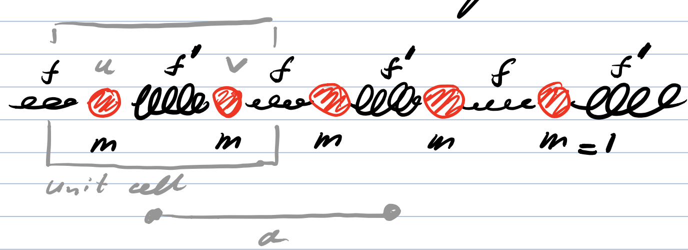
can dictate what happens on the surface.

7.3 Simple examples

7.3.1 The Su-Schrieffer-Heeger model

The goal of this section is to realize two things. First, we want to see how a bulk quantity is linked to an edge feature. Second, we try to understand how symmetries can by important.

Let us consider the following chain:



The equations of motion look like

$$-\omega^2 \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \underbrace{\begin{pmatrix} (f+f') & -f'-f e^{ika} \\ -f'-f e^{-ika} & (f+f') \end{pmatrix}}_{h(k)} \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

With $h(k) = f+f' - \underbrace{[f' + f\cos(k)]}_{d_x(k)} \sigma_x - \underbrace{f\sin(k)}_{d_y(k)} \sigma_y$

We see that $d_z(k) \equiv 0$ or equivalently

$$\{\sigma_z, h(k)\} = \sigma_z h(k) + h(k)\sigma_z = 0. \qquad (1)$$

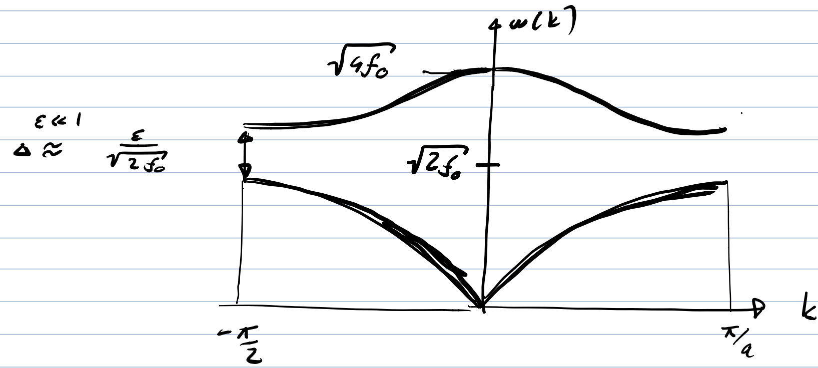
Eq (1) can be viewed as a symmetry constraint.

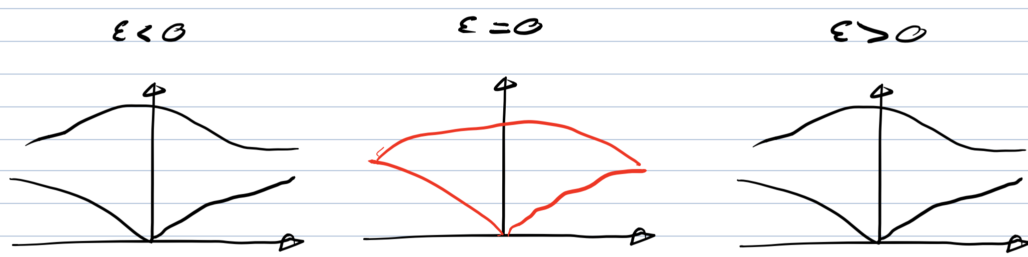
Let us write

$$f = f_0(1+\varepsilon)$$
$$f' = f_0(1-\varepsilon)$$

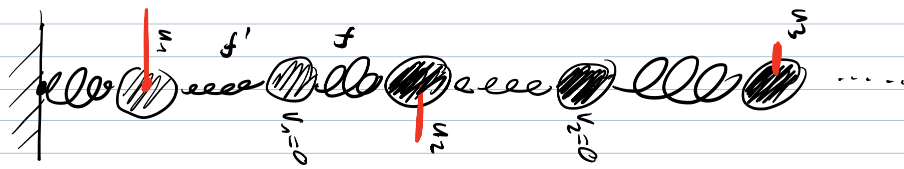
With this we obtain for the spectrum

$$\omega^2(k) = f_0\left[2 \pm \sqrt{1+\varepsilon^2+(1-\varepsilon^2)\cos(k)}\right]$$





We see that for $\varepsilon \neq 0$ we have two bands separated by a gap. Let us check what happens at a surface.



We find that we can construct a solution where there is no force on any of the v_i's and they stand still

$$m\ddot{v}_i = f'(u_i - v_i) - f(v_i - u_{i+1}) = 0 \qquad (2)$$

with $v_i \equiv 0$. We see that (2) requires

$$f' u_i = -f u_{i+1}.$$

This means that we have

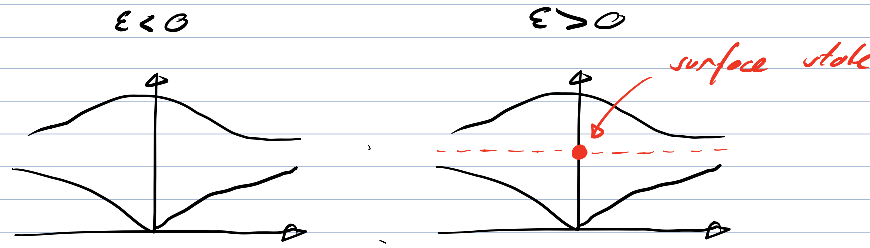
$$u_{i+1} = -\frac{f'}{f} u_i,$$

Which we solve with

$$u_i = e^{-r_i/\xi} \quad \text{with} \quad \xi = \log\frac{1+\varepsilon}{1-\varepsilon}.$$

What do we learn from that? We find an exponentially localized mode at the surface if $\xi > 0$. In other words if $\varepsilon > 0$! At $\varepsilon = 0$ $\xi \to \infty$ and also the band gap closes. We obtain the frequency of the edge state by solving

$$-\ddot{u}_i = -f'(u_i - v_i) - f(u_i - v_{i-1}) = -2f_0 u_i \Rightarrow \omega = \sqrt{2f_0}$$



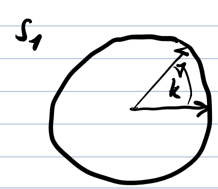
We see that the edge state lies at $\omega = \sqrt{2f_0}$, i.e., in the "middle" of the gap. We realize that we have a bandstructure that changes from $\varepsilon < 0$ to $\varepsilon > 0$ via a gap-closing and only on one side of this transition we have a surface state.

Bulk topology:

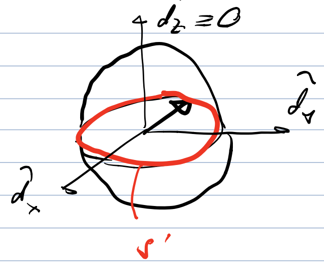
Is there something in the bulk that lets us predict this? Let us look at the evolution of the normalized $d(k)$-vector

$$\hat{d}(k) = \frac{\vec{d}(k)}{|\vec{d}(k)|}.$$

First of all, this can be done if $|\vec{d}(k)| \neq 0$ for all k-vectors. This is exactly the gap condition. Second, the Brillouin zone is a unit circle S^1.



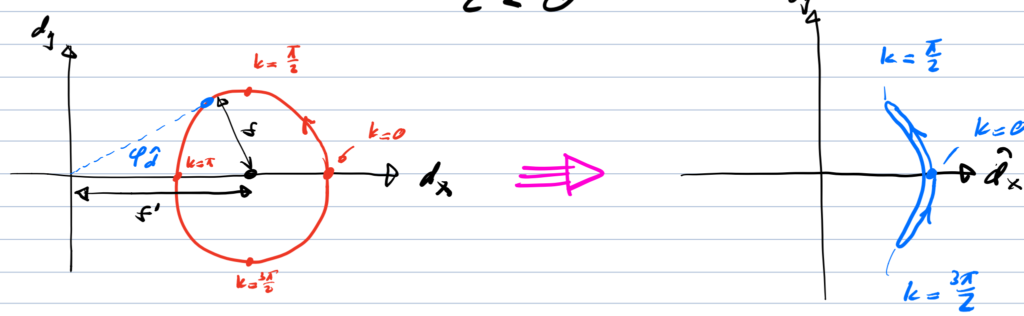
Moreover, as $\vec{d}$ has only two components, it is constrained to the equitorial plane:



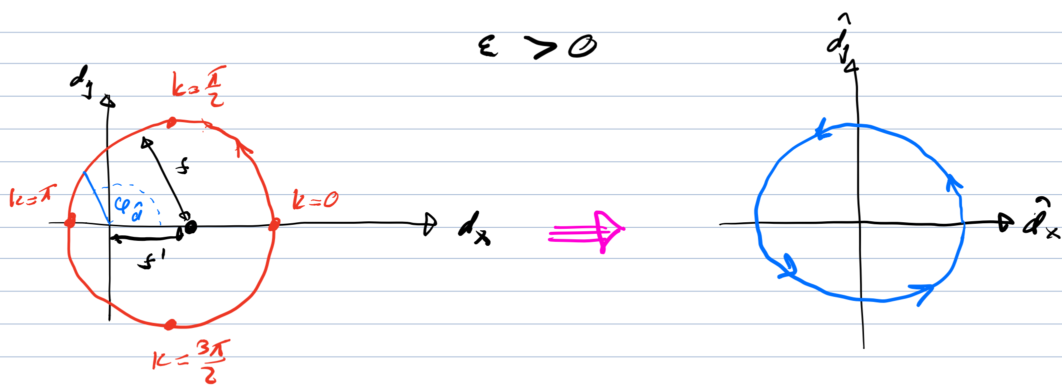
In other words, our equations of motion for the bulk define a mapping from $S^1 \to S^1$. Such mappings are characterized by a winding number: How many times is $\hat{d}(k)$ running around the unit circle if k goes around the unit circle once.

For our case we have

$\varepsilon < 0$



$\varepsilon > 0$



We observe that for $\varepsilon > 0$, $\hat{d}(k)$ indeed wraps once around the unit circle! The transition happens when the circle of $\vec{d}(k)$ touches the origin, exactly where the gap closes! We can cast this observation into a formula. The infitesimal increase in angle φ is given by

$$\omega(k) = e^{-i\varphi_k} \partial_k e^{i\varphi_k},$$

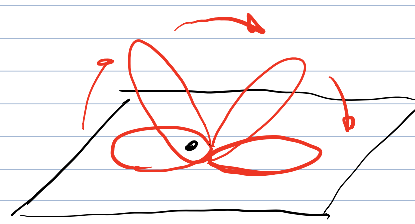
where we wrote the 2-dimensional, normalized d-vector as a complex number. The $\hat{d}$-vector now wraps S^1 m times with

$$C = \frac{1}{2\pi} \oint dk\, \omega(k).$$

The symmetry $\{\sigma_z, h(k)\}$ was important in the definition of this index! If the image of $[-\frac{\pi}{a}, \frac{\pi}{a}]$ can be anywhere on S^2, we can deform it without ever closing the gap:

With symmetry

Without symmetry



This model has recently been implemented in a phononic crystal, see references.

This model is the simplest example of the following corner-stones of topological crystals.

1.) An integer-valued index of the bulk can be defined. (winding number C)
2.) The value of this index makes a statement about the physics on the edge of the system. (localized mid-gap state)
3.) A symmetry is essential for the index to be well defined ($d_z \equiv 0$).

In the following we want to go to one dimension higher. In 2d, the edge of the system is one dimensional. This means, that possible edge states are not single isolated midgap states, but dispersing one dimensional channels.

7.3.2 Chern insulators

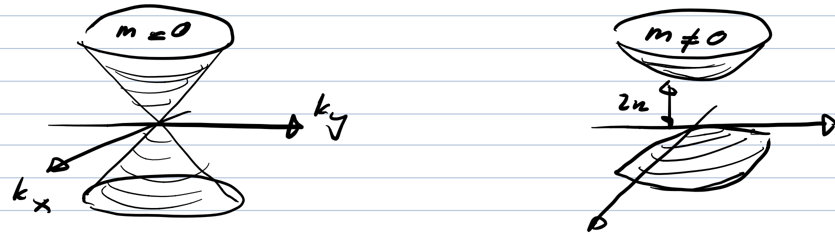
In this chapter we introduce the work-horse of two-dimensional topological system: the lattice Dirac-fermion Chern insulator. We will see that in its simple form, one cannot implement it in a passive meta-material. However, a large amount of our understanding is based on this example, which makes it a valuable model to know.

We start with simple Dirac fermions described by

$$H = \sum_{i=1}^{3} d_i(k)\,\sigma_i \quad \text{with} \quad \vec{d}(\vec{k}) = (k_x, k_y, m) \qquad (3)$$

We immediately see that the spectrum of H is given by

$$\omega(k) = \pm\sqrt{|k|^2 + m^2}$$

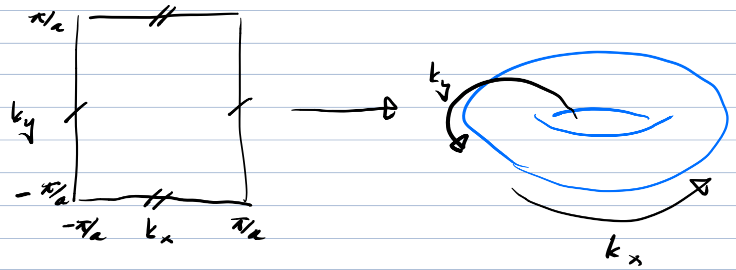


We immediately realize that H describes quite generically a touching $(m=0)$ or near touching $m \neq 0$ of two adjacent bands. However in its form (3) H does not describe a lattice system as it is not periodic in k_x, k_y. To turn it into a periodic model we simply write

$$(k_x, k_y, m) \longrightarrow [\sin(k_x), \sin(k_y), m + 2 - \cos(k_x) - \cos(k_y)] .$$

We see that for $k_x, k_y \to 0$ the two models are identical.

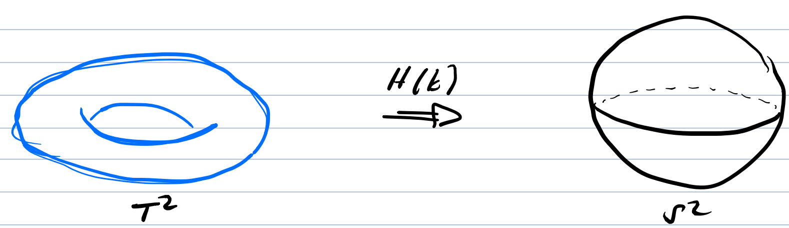
For the SSH model, the normalized $\vec{d}(k)$-vector encoded the bulk topology. What do we deal with in the present case? The first Brillouin zone has the structure of a torus T^2:



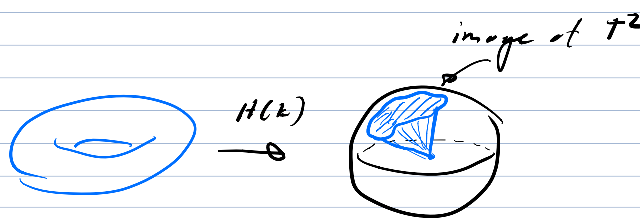
Here, however, we do not have any symmetry that constrains the d-vector and therefore the target space is the sphere S^2. In other words the family of matrices (one for each $\vec{k}$) defines a mapping

$$H(k): T^2 \to S^2$$

$$(k_x, k_y) \mapsto \hat{d}(k_x, k_y) = \frac{\vec{d}(k_x, k_y)}{|\vec{d}(k_x, k_y)|}.$$



What can now be the topologically distinct classes of Hamiltonians? It is the question if the $\hat{d}$-vector wraps the whole sphere!



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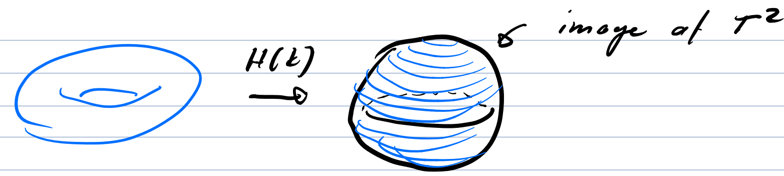
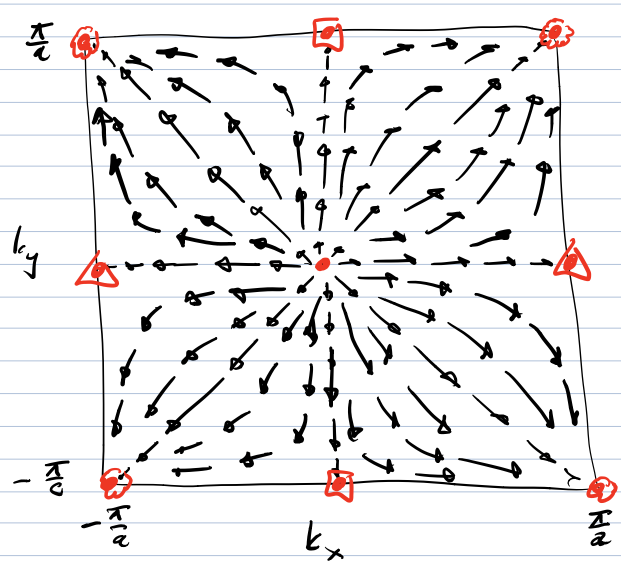


Clearly, these two situations are very different and not smoothly deformable into each other. Moreover, it is easy to see that this wrapping of S^2 is nothing but the generalization of our $S^1 \to S^1$ mapping in the SSH model! Note that the winding number for the SSH model could be both $\pm C$, depending on how the $\hat{d}$-vector rotated. The same is true here.

Let us look at our model. We first concentrate on the in-plane d-vector $[d_x(k), d_y(k)]$. This is like looking at S^2 from a star above the north-pole.



Note that (d_x, d_y) is zero only at the red points! $\Rightarrow$ gap closings can only happen there.

We see that certainly the $\hat{d}$ smoothly points in all directions on the equatorial plane. In order to know if we wrap to whole sphere, we need to check if we reach both south and north pole.

Let us start at $m < -4$. Then

$$d_z(k) = m + \underbrace{[2 - \cos(k_x) - \cos(k_y)]}_{\in [0,4]}$$

is positiv for all values of k_x, k_y and we never reach the south pole!

At $m = -4$, $d_z(\vec{k}) = 0$ at $(k_x, k_y) = (\pm\pi, \pm\pi)$. At this point also the in-plane part vanishes and therefore the gap is closing!

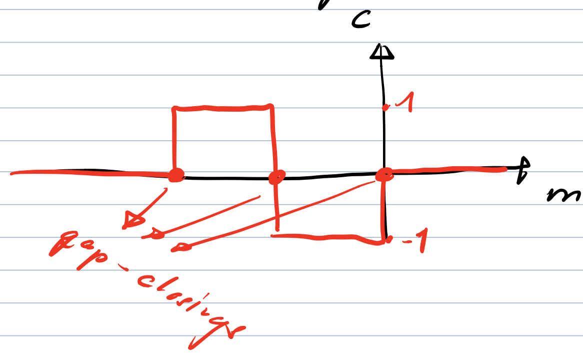
After we passed through $m = -4$ the normalized $\hat{d}$ is pointing to the north poles at $(0,0)$ and to the south-pole at $(\pm\pi, \pm\pi)$ {this is only one point!}.

$\Rightarrow$ We wrap the sphere once!

At $m = -2$ we close the gap at $(0, \pm\pi)$ and $(\pm\pi, 0)$. After that we still point both to the south and north-pole once, but we changed the orientation of the map (hard to see).

Finally, at $m=0$ we have a gap closing at $(0,0)$ and after that the $\vec{d}$-vector does not reach the north-pole anywhere in the Brillouine zone!

$\Rightarrow$ We have the following evolution of our index:



Of course this can be formalized by writing

$$C = \frac{1}{2\pi}\int_{T^2} d\vec{k}\ \varepsilon_{\alpha\beta\gamma}\ \hat{d}_\alpha\, \partial_{k_x} \hat{d}_\beta\, \partial_{k_y} \hat{d}_\gamma$$

which is know as the skyrmion number. In fact the skyrmion number is nothing but the special case of the Chern number if we only have two bands (remember, the H-matrix was 2×2). For a general problem we define

$$A_\alpha = i\langle \psi(k_x,k_y)|\,\partial_{k_\alpha}\,|\psi(k_x,k_y)\rangle$$

and $\quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad$ to give

$$C = \frac{1}{2\pi}\int_{T^2} d\vec{k}\ F_{\alpha\beta}$$

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