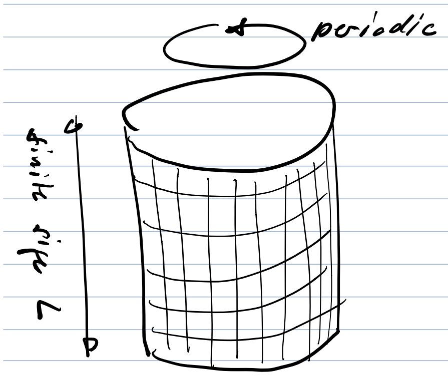
Edge states

We have now established two cornerstones of a topological band-structure: (i) a discrete topological index C, that (ii) can only change if we close a spectral gap.

Let us now check for the third property: non-trivial edge modes for $C \neq 0$. After all, it is these edge modes that we try to make use of for wave guiding applications.

For now, our model is written in $\vec{k}$-space. To check for edge-states, we want to study H on a cylinder.



This implies that we keep one direction, say x periodic and $e^{ik_x x}$ transports our solution in x-direction. However, in y direction we need to go to the direct space. Let us analyze what terms we deal with. First of all, we have a two-band model ($H = \vec{d} \cdot \vec{\sigma}$). Moreover, we now need to include L sites in y-direction $\Rightarrow$ H will be of the form

$$H = \begin{pmatrix} a & \gamma \\ \gamma^{+} & b \end{pmatrix},$$

where a describes the $L \times L$ matrix of internal stat a, b the $L \times L$ matrix of internal state b, and γ the mixing between them. In other words d_z encodes $a = -b$; d_x and d_y encode γ.

Let us see what terms we deal with in $d_z(k)$ we have

$$m + 2 - \cos(k_x) - \cos(k_y)$$

$\Rightarrow \pm[m+2-\cos(k_x)]$ stands on the diagonal of a (b) as these terms do not change the y-site [nothing transports us in y-direction]. However

$$-\cos(k_y) = -\frac{1}{2} e^{ik_y} - \frac{1}{2} e^{-ik_y}$$

encode a hopping by ± 1 in y-direction $\Rightarrow$

$$a = \begin{pmatrix} m+2-\cos(k_x) & -\frac{1}{2} & & \\ -\frac{1}{2} & m+2-\cos(k_x) & -\frac{1}{2} & \\ & -\frac{1}{2} & m+2-\cos(k_x) & -\frac{1}{2} \\ & & -\frac{1}{2} & \cdots \end{pmatrix}$$

and $b = -a$. We turn to $d_x(\vec{k}) = \sin(k_x)$. Clearly this just leads to the diagonal entry $\sin(k_x)$ on γ. $d_x(\vec{k})$ is harder.

$$\sin(k_y) = \frac{1}{2i}\left(e^{ik_y} - e^{-ik_y}\right)$$

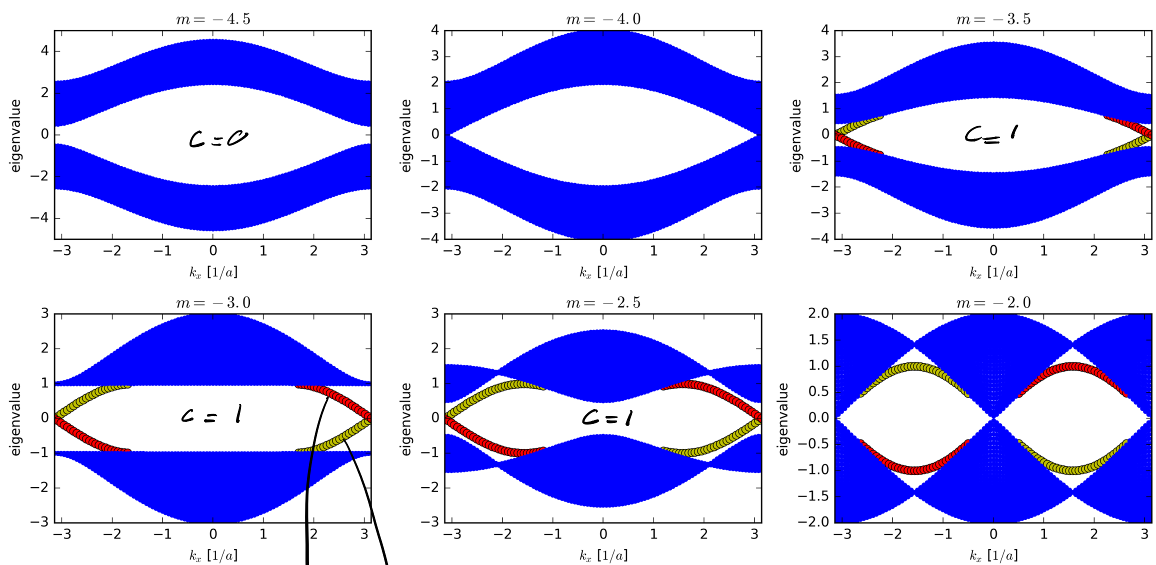
which connects again $y \pm 1$. Remember that $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, so to get γ we need to multiply this with $-i$

$$\Rightarrow \gamma = \begin{pmatrix} \sin k_x & -\frac{1}{2} & & & \\ \frac{1}{2} & \sin k_x & -\frac{1}{2} & & \\ & \frac{1}{2} & \sin k_x & -\frac{1}{2} & \\ & & \frac{1}{2} & \sin k_x & -\frac{1}{2} \\ & & & \frac{1}{2} & \cdots \end{pmatrix}$$

With this we can now diagonalize

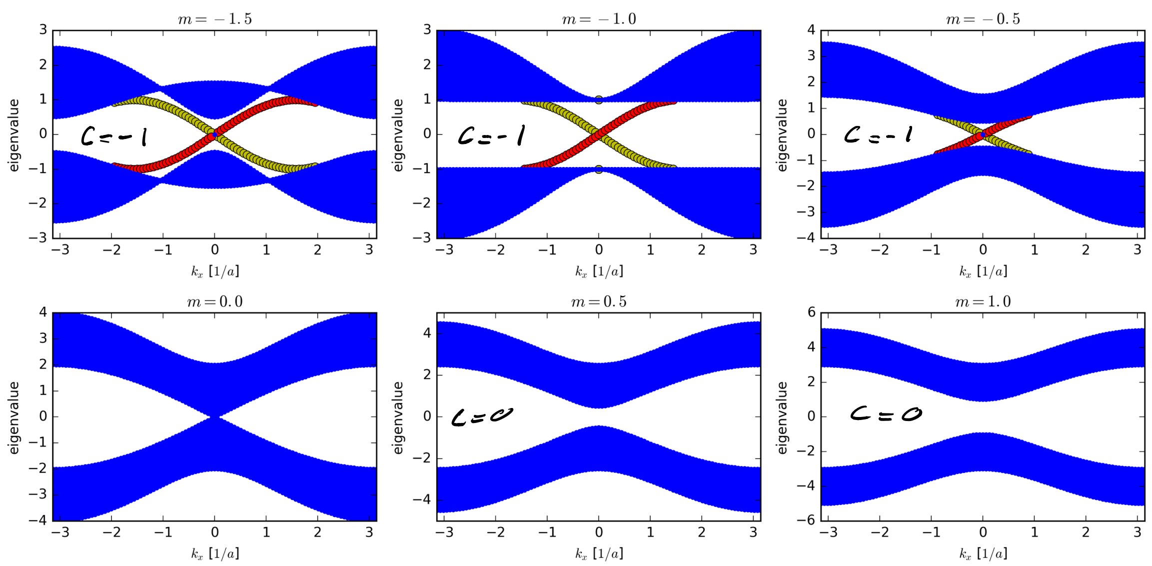
$$H = \begin{pmatrix} a & \gamma \\ \gamma^+ & -a \end{pmatrix}$$

to find:



yellow: bottom of cylinder

red: top of cylinder



7.3.3 Application in mechanics

We set out to establish a method to construct stable waveguides for vibrations. In its current form, H does <u>**not**</u> describe a valid dynamical matrix!

1.) Eigenvalues of H are smaller than zero.
$\Rightarrow$ in

$$\ddot{\vec{x}} = H\vec{x} \rightsquigarrow -\omega^2 \vec{x} = H\vec{x}$$

we will have to take the square root of negative eigenvalues

$\Rightarrow$ this can be cured by adding

$$\mu \mathbb{1}$$

to H with μ large enough.

2.) We have terms of the form

$i\sin(k_y)$ and $\sin(k_x)$

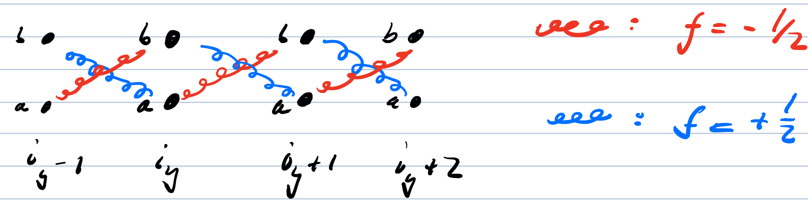
which become

$$\ddot{a}_{i_x,i_y} = -\frac{1}{2} b_{i_x,i_y+1} + \frac{1}{2} b_{i_x,i_y-1}$$

etc.., and

$$\ddot{a}_{i_x,i_y} = \frac{i}{2} b_{i_x-1,i_y} - \frac{i}{2} b_{i_x+1,i_y}$$

The first problem is in principle doable:



The second problem we cannot implement with a mass spring model as we would need complex springs.

This observation tells us that we cannot have truly unidirectional channels in <u>passive</u> mass-spring systems.

In a last part on topological metamaterials we will learn how to circumvent this problem!

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