7.4 Time reversal invariant topological systems

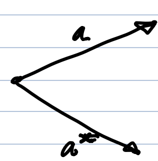
We have seen that a simple Chern insulator with $\vec{d}(k) = (\sin k_x, \sin k_y, m+2-\cos k_x - \cos k_y)$ necessarily leads to couplings between modes which is imaginary. While for electrons in solids this is a valid possibility, such a coupling cannot be implemented with mechanical degrees of freedom. To cure this problem we have two options. (i) We can go beyond the simple equation of motion in the form of

$$\ddot{\vec{x}} = D \vec{x}$$

by using gyroscopes. (ii) A second variant is to take a Chern insulator and double the degrees of freedom in a suitable manner. We start with the second strategy.

7.4.1 Doubling the degrees of freedom.

We had a system H which encodes imaginary couplings. One way to turn a complex number real, is by adding its complex conjugate



$$\Rightarrow a + a^* \in \mathbb{R}.$$

We follow this idea by doubling the degrees of freedom by writing:

$$\mathcal{D} = \begin{pmatrix} H & 0 \\ 0 & H^* \end{pmatrix}.$$

Let us see what happens if we transform to a new basis

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \Rightarrow$$

$$\tilde{\mathcal{D}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & H^* \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} H & -Hi \\ H^* & H^* i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(H+H^*) & \frac{i}{2}(H-H^*) \\ -\frac{i}{2}(H-H^*) & \frac{1}{2}(H+H^*) \end{pmatrix}$$

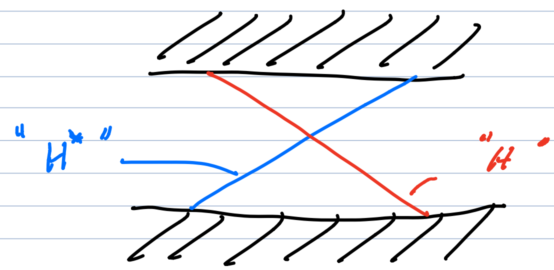
$$= \begin{pmatrix} \mathrm{Re}\, H & \mathrm{Im}\, H \\ -\mathrm{Im}\, H^* & \mathrm{Re}\, H \end{pmatrix} \in \mathbb{R}!$$

We found a way to render all couplings real by doubling the degrees of freedom.

Note that the Chern number for H^* is the negative of the Chern number for H:

$$C^* = -C.$$

This implies that our new edge spectrum looks with



This now implies that our surface states are stable, if and only if, there are no terms mixing the two sectors in a way that breaks time reversal symmetry. We therefore deal with a symmetry protected topological state. This is analogous to the situation for the SSH chain discussed above!

7.4.2 The doubled $\frac{1}{3}$-flux model

Doubling the Chern insulator discussed above does not lead to a simple mechanical model (why?). We present here a simpler version corresponding to electrons hopping on a lattice in the presence of a magnetic field. The model writes

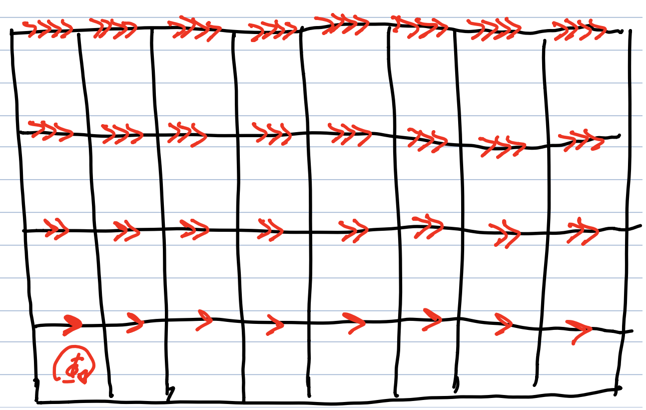
Hopping with $e^{i\varphi}$, where

$\varphi = 0$ for

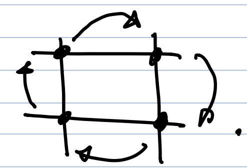
$\varphi = \Phi_0$ for

$\varphi = 2\Phi_0$ for

etc. And if we hop against the arrow, we

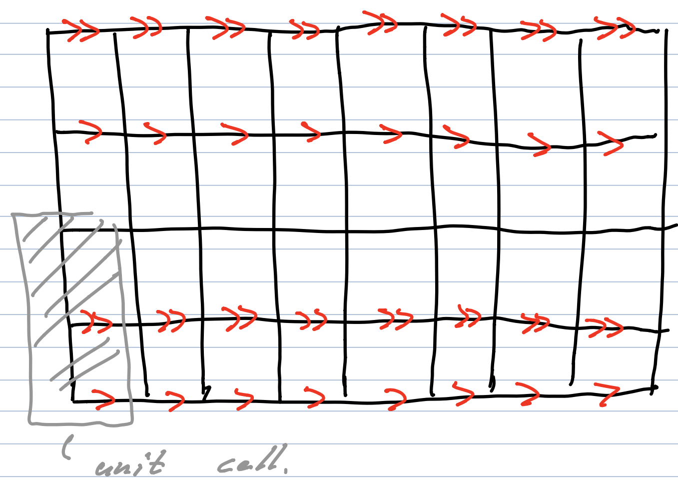


pick up $e^{-i\varphi}$. Note, that for our phase pattern, the total "flux" picked up when hopping around a plaquette is Φ_0. Note the difference to the wave number $\vec{k}$, where we would pick up $ik_x - ik_y - ik_x + ik_y = 0$ for a round trip of the form

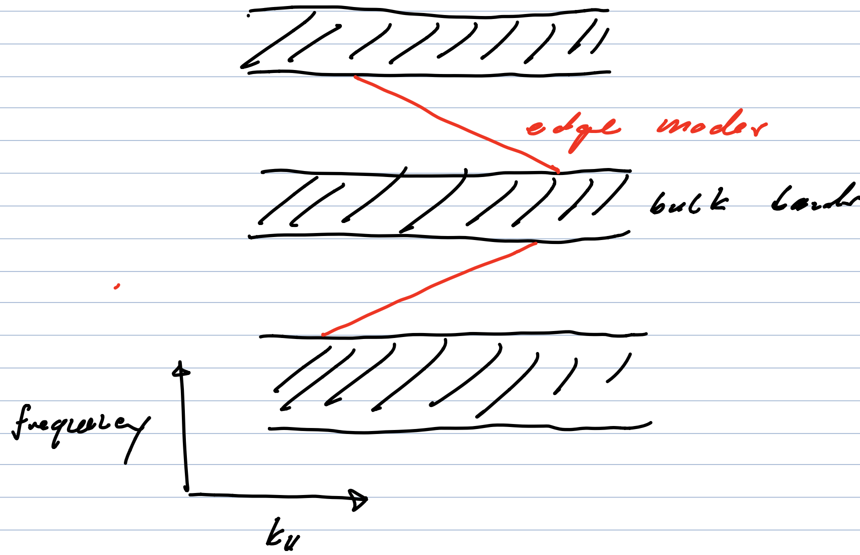


This non-vanishing flux Φ_0 corresponds to a magnetic field $B = \frac{1}{ec}\Phi_0 \cdot a^2$, where a is the lattice constant.

We specialize to $\Phi_0 = \frac{2\pi}{3}$ and hence find



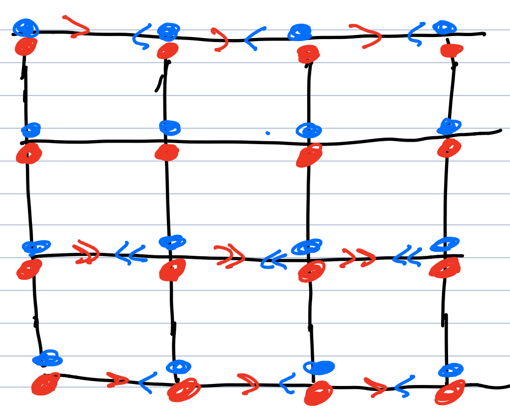
One sees that we have a unit cell with three sites and the spectrum on a cylinder has three bands and both band gaps have a non-vanishing Chern number $C = \pm 1$



This model now constitutes H, which has obviously imaginary entries. To construct the full model we write

$$\mathcal{D} = \begin{pmatrix} H & \\ & H^* \end{pmatrix}$$

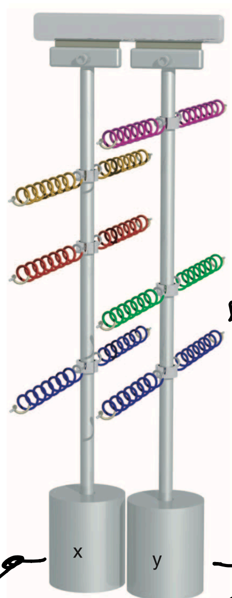
or



Using the above logic we obtain a good dynamical matrix via

$$\begin{pmatrix} x_{r,s} \\ y_{r,s} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} {\color{blue}\bullet}_{r,s} \\ {\color{red}\bullet}_{r,s} \end{pmatrix}$$

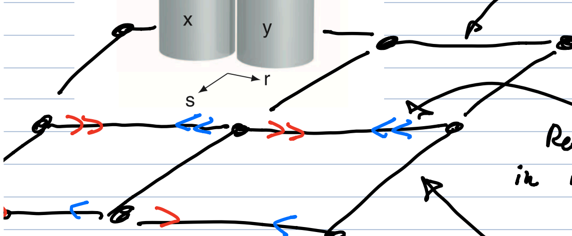
on every site. These degrees of freedom can now be implemented in the mechanical domain! We need two (x and y) degrees per site which we implement with pendulums



They can only swing in one direction, i.e., each of them makes one mode.

the springs are attached to couple to neighboring sites.

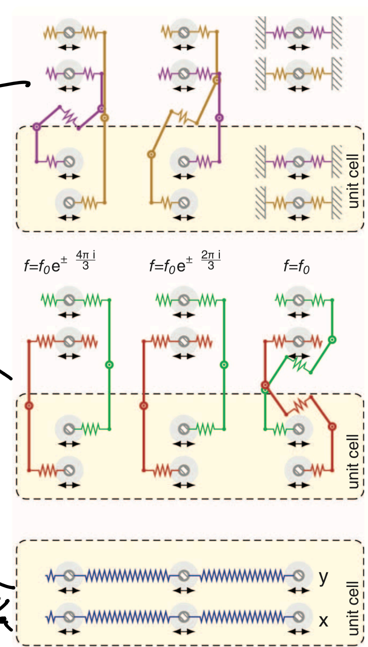
Im H in r-dir.



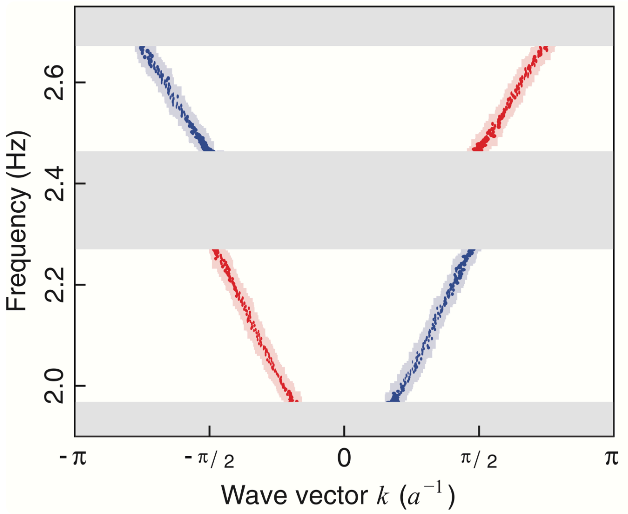
Re H in r-direction

coupling in r-direction

Im H = 0



For a 9×15 lattice with a total of 270 pendulums the resulting measured spectrum looks like



A few comments

(i) We indeed find counterpropagating modes that differ by "color"

(ii) $\propto x + iy$ and $\propto x - iy$. In other words, interpreted as a two-dimensional pendulum with coordinates $(x(t), y(t))$, the two modes are the two circularly polarized modes!

(iii) By building a transducer that couples only to left or right circular polarization, one can turn the edge channels unidirectional.

7.5 Gyroscope systems.

For setups implementing true Chern insulators with mechanical degrees of freedom, we refer to

Exp:	Nash et al.	PNAS	**112**	14495	(2015)
Theory:	Wang et al.	PRL	**115**	104302	(2015)
	Süsstrunk & SDH	PNAS	**113**	E4767	(2016)

References

1. V. Klitzing, K., Dorda, G. & Pepper, M. "New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance". *Phys. Rev. Lett.* **45,** 494 (1980).
2. Thouless, D. J., Kohmoto, M., Nightingale, M. & den Nijs, M. "Quantized Hall Conductance in a Two-Dimensional Periodic Potential". *Phys. Rev. Lett.* **49,** 405 (1982).
3. Kane, C. L. & Mele, E. J. "Quantum Spin Hall Effect in Graphene". *Phys. Rev. Lett.* **95,** 226801 (2005).
4. Brüne, C. *et al.* "Quantum Hall Effect from the Topological Surface States of Strained Bulk HgTe". *Phys. Rev. Lett.* **106,** 126803 (2011).
5. Xiao, M. *et al.* "Geometric phase and band inversion in periodic acoustic systems". *Nature Phys.* **11,** 240 (2015).
6. Süsstrunk, R. & Huber, S. D. "Observation of phononic helical edge states in a mechanical topological insulator". *Science* **349,** 47 (2015).
7. Süsstrunk, R. & Huber, S. D. "Classification of topological phonons in linear mechanical metamaterials". *Proc. Natl. Acad. Sci. USA* **113,** E4767 (2016).
8. Nash, L. M. *et al.* "Topological mechanics of gyroscopic metamaterials". *Proc. Natl. Acad. Sci. USA* **112,** 14495 (2015).
9. Wang, P., Lu, L. & Bertoldi, K. "Topological Phononic Crystals with One-Way Elastic Edge Waves". *Phys. Rev. Lett.* **115,** 104302 (2015).