Chapter 8

Maxwell frames

Learning goals

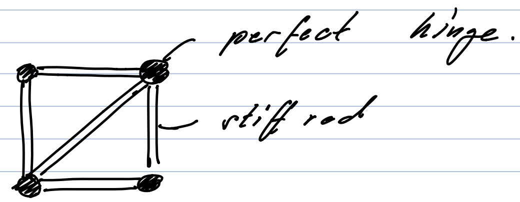
- You know what a Maxwell frame is.
- You know compatibility and equilibrium matrices.
- You know the significance of a state of self-stress.

Large parts of this chapter are based on the review by Lubensky et al. [1].

8. Maxwell frames

Up to now we were mainly concerned with meta-materials that control the flow of waves at finite frequency. In this chapter we wants to introduce a setting which is tailored towards the control of zero-frequency properties.

There are a large number of techniques to shape the elastic zero-frequency response. Here, we discuss the use of framed structures, i.e., collections of (stiff) rods connected by (perfect) hinges



The choice for this system is motivated by several aspects. First, such frames are used in structural mechanics, robotics, material sciences, etc. and they have a simple but beautiful mathematical description.

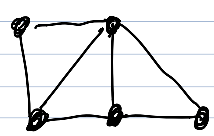
James Clerk Maxwell realized that for a frame to be "stiff", i.e., a system where the distance between hinges cannot be altered without changing the length of any rod, it requires

$$N_R = dN - f(d) \qquad (1)$$

rods. Here $f(d) = d(d+1)/2$ is the number of independent rigid translations and rotations and N is the number of points, or hinges. It is clear that ever point brings d degrees of freedom. Every single bond, on the other hand constrains one degree of freedom. As rigid translations and rotations do not affect the stiffness we obtain (1).

We are now interested in so-called Maxwell frames, where we are exactly at the threshold to stiffness. Why is this interesting? If we want to design certain mechanical functionalities, we profit from sitting right at threshold, such that the controlled removal or addition of bars or other constraints yields the targeted behavior.

To continue, we must realize that (1) is incomplete. Consider the following frame

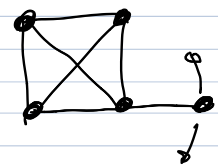


$N = 5$

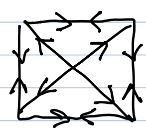
$N_R = 7$

$\Rightarrow 7 = 2 \times 5 - 3$

and we would conclude, rightfully, that this frame is stiff. Let us change this frame slightly:



Clearly, the number of bars and points did not change! But the rightmost point is not stiff! What went wrong? We added a redundant bar to the alredy stiff square. What this bar introduced is a state of self stress: In a state of self stress there can be stresses on the bars without net forces on the hinges.



Example of a state of self stress.

These states of self stress play an important role for the stiffness: load bearing frames need in two dimensions three states of self-stress (x, y, and shear load).

The generalized Maxwell counting takes this duality between zero modes (where points can move without a restoring force) and states of self-stress into account

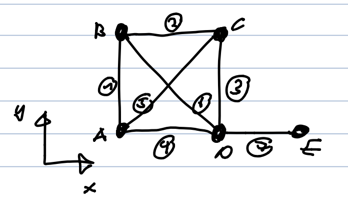
$$\boxed{N_0 - N_s = dN - N_R} .$$

8.1 Compatibility and equilibrium matrices

To formalize the theory of Maxwell frames we introduce two matrices Q and C. Q, the equilibrium matrix relates bond tensions T to forces F on the hinges

$$-F = QT.$$

Let us construct it for



$$
-\begin{pmatrix} F_A^x \\ F_A^y \\ F_B^x \\ F_B^y \\ F_C^x \\ F_C^y \\ F_D^x \\ F_D^y \\ F_E^x \\ F_E^y \end{pmatrix}
=
\underbrace{\begin{pmatrix}
0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 & 0 \\
1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
-1 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & -1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & -1 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -\frac{1}{\sqrt{2}} & 1 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}_{Q}
\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \end{pmatrix}
$$

Q has $\operatorname{rank} Q = 5$ with $t_0 = (1, 1, 1, 1, -\sqrt{2}, -\sqrt{2}, 0)$ as the null-vector of Q, i.e., our state of self-stress.

The compatibility matrix C connects displacements of points U to bond elongations. E.

$$E = CU.$$

It is easy to see that $C = Q^T$. Moreover, the null-space of C contains the two rigid translations and the rigid rotation as well as

$$u_0 = (0,0,0,0,0,0,0,0,0,1)$$

the zero mode of the right-most point. We see how states of self-stress and zero modes appear as dual partners through C and Q.

As a side note, so far we only considered dynamical matrices D, connecting points with points, neglecting the degree of freedom of the rods. However, they are of course not independent but related by

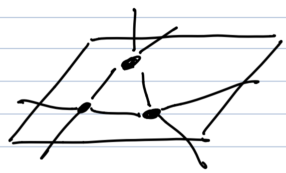
$$D = C^T \begin{pmatrix} f_1 & & & \\ & f_2 & & \\ & & \ddots & \\ & & & f_e \end{pmatrix} C,$$

where f_i denotes the spring constant describing the i'th rod. Note that C brings us from points to bars and $C^T = Q$ back to points, as it should be. We can therefore think of C as the square root of D.

Note, that for Maxwell frames with $dN = N_R$, both D and C are square matrices of size $dN \times dN$.

8.3 Periodic Maxwell frames

Let us now specialize to periodic Maxwell frames, where we have a unit cell with M points that repeats.



For the frame to be critically coordinated we need

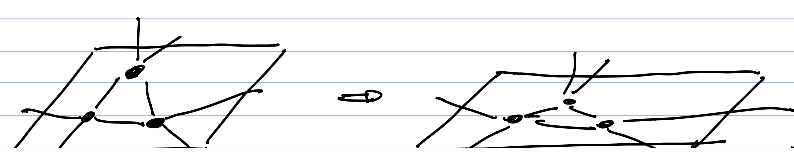
$$dN = N_B$$

per unit cell. For such periodic, critically coordinated frames there is a peculiarity called Guest modes.

8.3.1 Guest modes

Let us assume that a 2D frame is rigid. ⇒ We need three states of self stress (x, y, shear). In other words the dimension of the null space of C^T is (at least) 3.

This immediately implies that also C has a kernel of size 3 as we have square matrices owing to $dN = N_B$. For finite frames we would immediatly say that these 3 zero modes are the two translations and the rotation. Here, however, rotations are not allowed by the requirement that the frame is periodic! ⇒ There is one extra zero-mode called the "Guest" mode: It corresponds to a change in the unit-cell shape

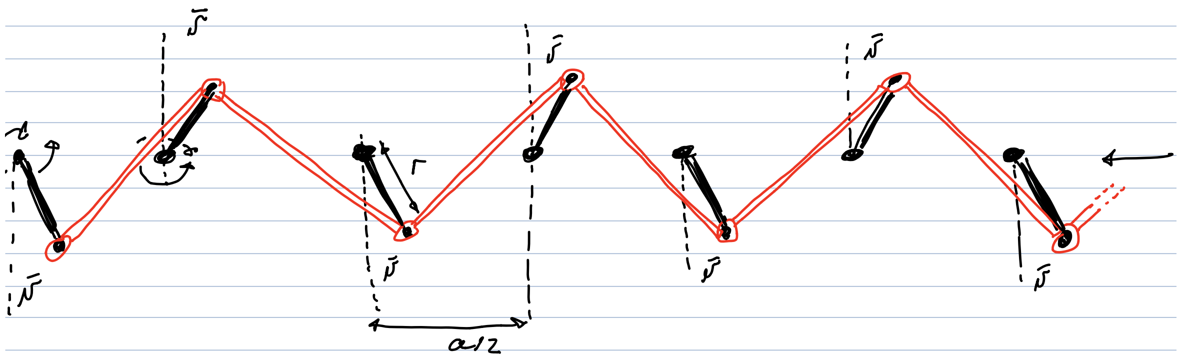


In other words the structure is <u>not</u> rigid after all!
While this argument is based on C and Q and therefore only valid for infinitesimal changes in a, T, or ξ, one can readily generalize it: the structure after an infinitesimal change of the unit-cell is again (i) critically coordinated ($dN = N_E$) and (ii) periodic. Hence we can repeat the argument!

8.3.2. The Kane-Lubensky theory

In 2014 Kane and Lubensky made the interesting discovery, that Maxwell frames can have a structure similar to the topological band theory that we discussed, but on C rather than $D = C^T C$. In particular, they showed how one can use a certain winding number to describe both states of self-stress and zero modes. [Note, that states of self-stress disappear from the description once we move to $D = C^T C$].

We consider a simple example in one dimension



All black bars are fixed on a hinge on the middle line indicated by ← above. At their ends they are connected by (red) rods. Every point hence have one degree of freedom, the angle ϑ. Each red bar induces a constraint, and we see that we indeed have a critically coordinated frame.

The compatibility matrix C now connects stretches of the bars δl_β with rotations $\delta\vartheta_s$

$$\delta l_\beta = C_{\beta s}\,\delta\vartheta_s$$

The compatibility matrix depends on the rest angle $\bar{\vartheta}$ as

$$C_{\beta s}(\bar{\vartheta}) = c_1(\bar{\vartheta})\,\delta_{s,\beta} - c_2(\bar{\vartheta})\,\delta_{s,\beta+1}$$

with

$$c_{1,2} = \frac{(a \pm r\sin\bar{\vartheta})\,r\cos\bar{\vartheta}}{\sqrt{a^2 + 4r^2\cos^2\bar{\vartheta}}}$$

The Fourier transform of $C_{\beta s}$ is given by

$$C(q) = c_1 - c_2\,e^{iqa}$$

This form immediately reminds us of the SSH model discussed above. Note, however, that here we consider C rather than $C^T C$. Indeed

$$C^T C = \begin{pmatrix} c_1 & & & & \\ -c_2 & c_1 & & & \\ & -c_2 & c_1 & & \\ & & -c_2 & c_2 & \\ & & & -c_2 & \ddots \end{pmatrix} \begin{pmatrix} c_1 & -c_2 & & & \\ & c_1 & -c_2 & & \\ & & c_1 & -c_2 & \\ & & & c_1 & -c_2 \\ & & & & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} c_1^2 & -c_1 c_2 & & & \\ -c_1 c_2 & c_1^2+c_2^2 & -c_1 c_2 & & \\ & -c_1 c_2 & c_1^2+c_2^2 & -c_1 c_2 & \\ & & -c_1 c_2 & c_1^2+c_2^2 & -c_1 c_2 \\ & & & & \ddots \end{pmatrix}$$

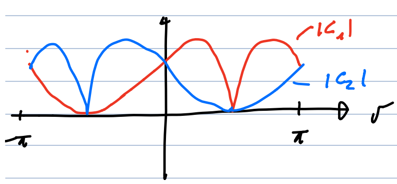
is just a regular hopping model without any "topological" structure.

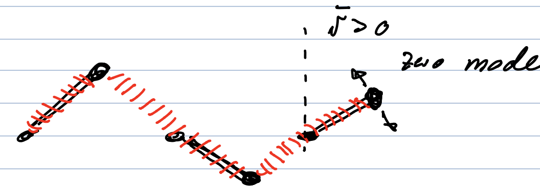
Let us go back to C. If we cut open the chain we remove one bar and should therefore get one zero mode. Its form can immediately determined by

$$c_1 - \lambda c_2 = 0 \quad \Rightarrow \quad \lambda = \frac{c_1}{c_2} = e^{-\varkappa a} \quad (e^{\varkappa a})$$

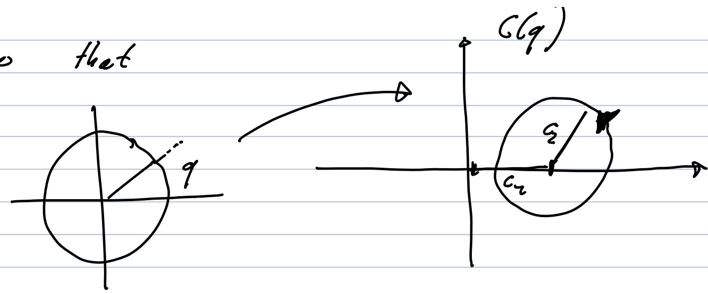
for states localized on the left (right). Therefore, the one zero mode is at the left if $|c_1| < |c_2|$ and at the right if $|c_1| > |c_2|$.

This mode is easy to understand at $\bar{\vartheta}_c = -\sin^{-1}\left(\frac{a}{2r}\right)$ where $c_1 = 0$:





We know that

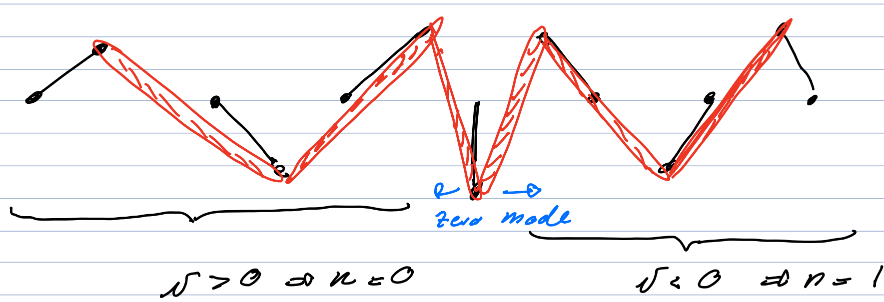


is characterized by a winding number

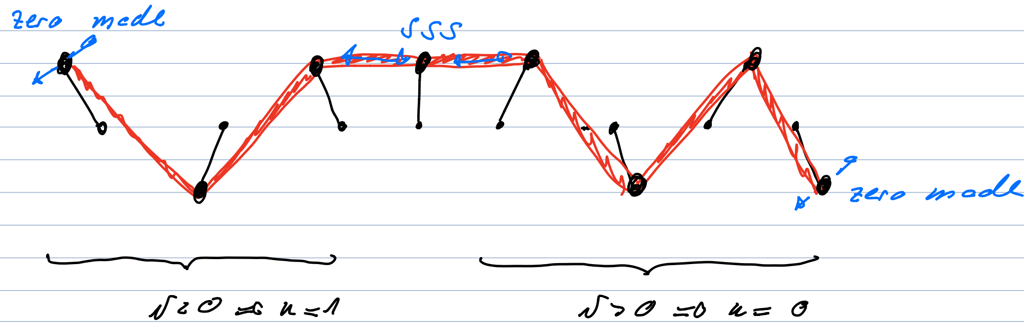
$$n = \frac{1}{2\pi} \oint d\varphi = \frac{1}{2\pi i} \int_0^{2\pi/a} dq \, \frac{\partial}{\partial q} \, \mathrm{Im} \left[\log \left[\det C \right] \right].$$

which in our case is $+1, 0$ depending on δ. Comparing with the form of the surface state, we realize that for $n=1$ the edge state is at the left, whereas for $n=0$ it sits at the right.

We can summarize that if $n=0$ we have one more zero mode on the left than on the right. What happens if we make a domain wall:

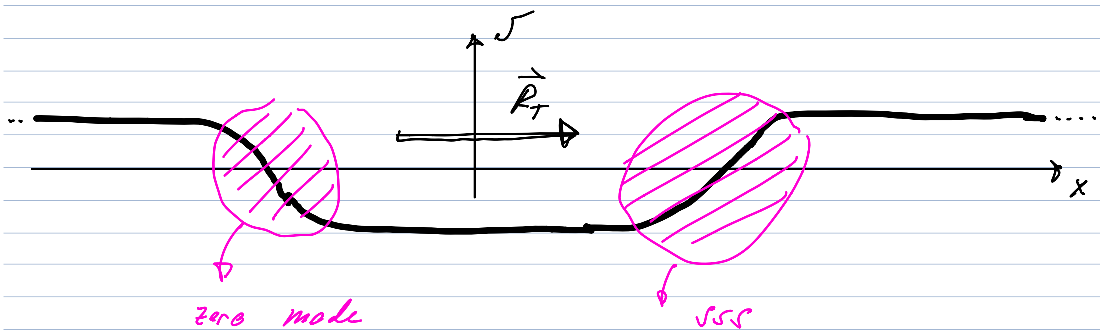


or vice versa



We find, as it should be, that #zero modes − #SSS = 1. However, stacking two systems of opposite topological polarization either creates a zero mode at the boundary or a SSS.

In a infinite chain one can think of creating,



from which we see why one calls n a topological polarization: It indicates how the two dual partners (SSS's and zero modes) are separated in an otherwise rigid lattice.

This polarization can now be used to construct deliberately zero modes or SSS, which can be helpful for robotics (zm) or design failure points (SSS).

References

1. Lubensky, T. C., Kane, C. L., Mao, X., Souslov, A. & Sun, K. "Phonons and elasticity in critically coordinated lattices". *Rep. Prog. Phys.* **78,** 109501 (2015).
2. Guest, S. D. & Hutchinson, J. W. "On the determinacy of repetitive structures". *J. Mech. Phys. Solids* **51,** 383 (2003).
3. Kane, C. L. & Lubensky, T. C. "Topological boundary modes in isostatic lattices". *Nature Phys.* **10,** 39 (2013).