Chapter 1

Bragg scattering vs. local resonances

Learning goals

- You can formulate the principle of Bragg scattering.
- You can relate the dimension of a phononic crystal to the frequency of a potential band gap.
- You know the effect of the coupling of two near-identical oscillators.

1.1 Bragg scattering vs. local resonances

We start our lecture on mechanical metamaterials with the discussion of metamaterials with a target functionality at *finite frequencies*. In other words, we want to understand how we can control or manipulate the flow of mechanical energy in the form of vibrations, or equivalently waves.
To understand how vibrations propagate through materials we will have to introduce the wave equation and familiarize ourselves with its properties. However, before we do so, we introduce two generic building blocks that are used to achieve the goal of manipulating wave propagation in solids: Bragg scattering and the use of locally resonant structures. Once we know how these two principles work, we will understand most modern metamaterial designs.

1.1.1 Bragg scattering

For now, all we need to know about waves is that they can be described by a traveling modulation of some property $f(x,t)$ such as

$$f(x,t) = \cos(\omega t - kx + \varphi), \tag{1.1}$$

where $\omega = 2\pi/T$ is the angular frequency related to the period T, $k = 2\pi/\lambda$ is the wave number related to the wave-length λ, and φ is a phase.

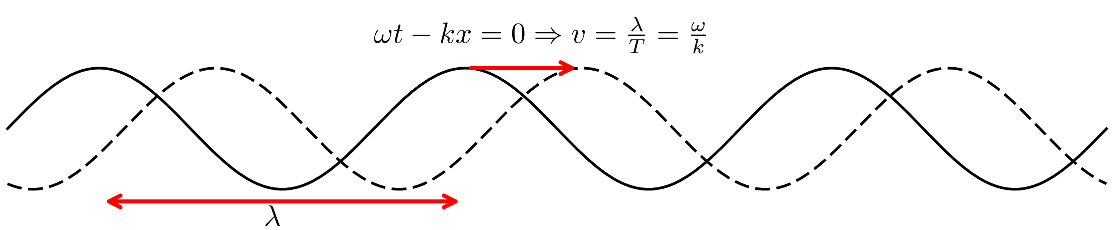
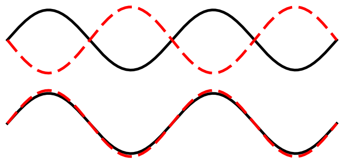


Figure 1.1: Traveling waves.

One of the most important aspects of waves is their ability to interfere. In particular:



$\Rightarrow$ destructive interference for $\Delta\varphi = \pi$.

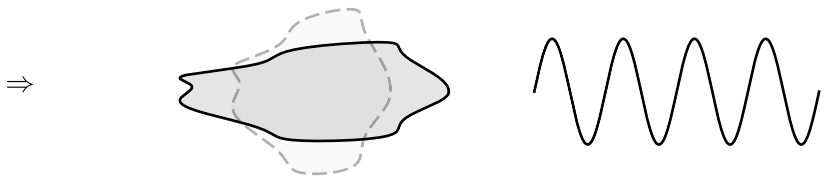
$\Rightarrow$ constructive interference for $\Delta\varphi = 0$.

Let us now consider a wave incident on a scatterer:

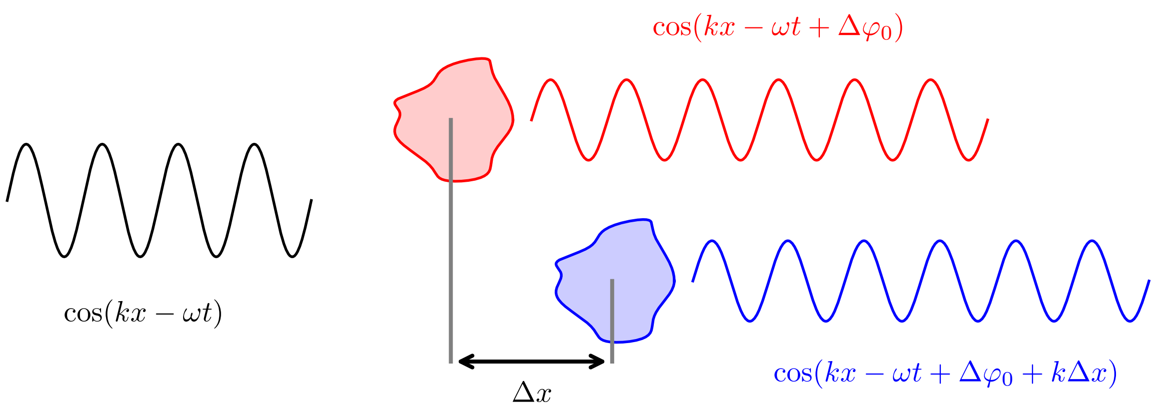
1.) Wave is incident on scatterer

2.) Scatterer is excited

3.) Scatterer radiates waves



The "time lag", i.e., the phase difference $\Delta\varphi$, between the incident and the radiated wave is given by the details of the interaction between the wave and the scatterer and the properties of the excitation of the scatterer. In other words, we have nothing too generic to say about this event. However, consider the effect of two *identical* scatterers at a distance Δx:



We see that for

$$
\begin{aligned}
k\Delta x = \pi \quad &: \text{the scattered waves interfere destructively.} \\
\Rightarrow \Delta x = \frac{\lambda}{2} \quad &: \text{leads to a strong destructive effect on the propagation of waves.}
\end{aligned}
$$

This phenomena is called "Bragg scattering" if we deal with *periodic arrays* of scatterers. Bragg scattering is at the heart of phononic crystals. It is not hard to imagine that periodic arrays with a typical length-scale ΔL can have dramatic effects on the wave propagation at frequencies where the Bragg condition

$$\Delta L = \frac{\lambda}{2} \tag{1.2}$$

is fulfilled, see Fig. 1.2.

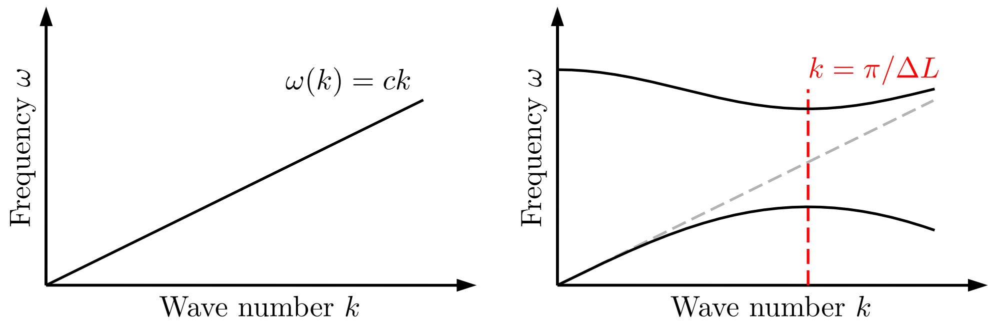


Figure 1.2: Band gap of a phononic crystal.

If I want to change the propagation of waves at a frequency $\nu = \omega/2\pi$, I have to design structures with dimension $\Delta L \approx c/\nu$.

In particular, at the frequencies where the Bragg condition is met, a window opens where no waves can propagate. We call this a band gap.
Question: I live in California and want to protect my house from earthquakes with a phononic crystal. How big is it going to be?

1.1.2 Local resonances

We have seen that periodic structures can have an influence on the propagation of waves. Here, we want to present a building block to achieve a similar effect, however, without the limit imposed by the size of the structure. In order to understand the concept of local resonances, we need to understand the coupling of two oscillators.
Let as assume the following system illustrated in Fig. 1.3:
The oscillators $x(t)$ and $y(t)$ are described by

$$\ddot{x}(t) = -\omega_0^2 x(t) + \gamma^2 y(t), \tag{1.3}$$
$$\ddot{y}(t) = -\omega_0^2 y(t) + \gamma^2 x(t), \tag{1.4}$$

where we want to assume that $\gamma \ll \omega_0$. For $\gamma = 0$ we know that

$$x(t) = x_0 e^{i\omega_0 t} \quad \text{and} \quad y(t) = y_0 e^{i\omega_0 t}, \tag{1.5}$$

solve the equations (1.3) and (1.4). If the two oscillators are coupled ($\gamma \neq 0$), we have to find combined solutions

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \quad \Rightarrow \tag{1.6}$$
$$-\omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -\omega_0^2 & \gamma^2 \\ \gamma^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{1.7}$$

In other words, we are dealing with a eigenvalue problem for ω. We have two routes to solve this problem.
Route 1: We solve the characteristic equation:

$$\det \begin{pmatrix} \lambda - \omega_0^2 & \gamma^2 \\ \gamma^2 & \lambda - \omega_0^2 \end{pmatrix} = (\lambda - \omega_0^2)^2 - \gamma^4 = 0 \tag{1.8}$$
$$\Rightarrow \quad \lambda_\pm = \omega_0^2 \pm \gamma^2 \quad \Rightarrow \quad \omega_\pm = \sqrt{\omega_0^2 \pm \gamma^2}. \tag{1.9}$$

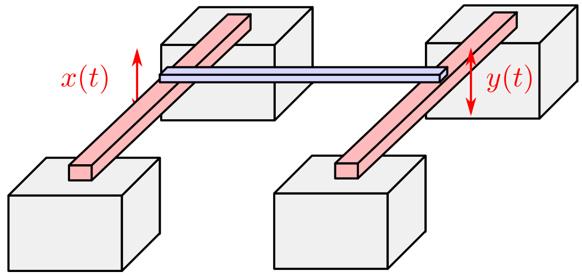


Figure 1.3: Two coupled oscillators.

Route 2: Basically the same, but once and for all:

$$\begin{pmatrix} -\omega_0^2 & \gamma^2 \\ \gamma^2 & -\omega_0^2 \end{pmatrix} = -\omega_0^2 \mathbb{1} + \gamma^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.10}$$

$$= -\omega_0^2 \mathbb{1} + \sum_{i=1}^{3} d_i \sigma_i \qquad \text{with} \tag{1.11}$$

$$\mathbf{d} = (\gamma^2, 0, 0) \qquad \text{and} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.12}$$

From this we immediately read

$$\lambda_\pm = -\omega_0^2 \pm |\mathbf{d}| = -\omega_0^2 \pm \gamma^2. \tag{1.13}$$

We can now easily generalize that to two coupled oscillators that have different frequencies:

$$\begin{pmatrix} -\omega_x^2 & \gamma^2 \\ \gamma^2 & -\omega_y^2 \end{pmatrix} = -\frac{\omega_x^2 + \omega_y^2}{2} \mathbb{1} + \gamma^2 \sigma_1 - \frac{\omega_x^2 - \omega_y^2}{2} \sigma_3 \quad \Rightarrow \quad \mathbf{d} = \left(\gamma^2, 0, -\frac{\omega_x^2 - \omega_y^2}{2} \right). \tag{1.14}$$

And the new eigenfrequencies are

$$\omega_\pm = \sqrt{\left(\frac{\omega_x^2 + \omega_y^2}{2} \right)^2 \pm \left[\left(\frac{\omega_x^2 - \omega_y^2}{2} \right)^2 + \gamma^2 \right]^2}. \tag{1.15}$$

As illustrated in Fig. 1.4a, the effect of a coupling between two oscillators is strongest if they are degenerate, i.e., have the same frequency. Moreover, the main effect of γ is to *split the frequency of the two-oscillator system*. In Fig. 1.4b & c we see what effect we can expect for a local resonance onto the dispersion of waves: The coupling will split the degenerate system of the wave and the local oscillator. This effect is strongest where the two frequencies cross. Again, like in the Bragg scattering effect, a frequency window is opening where no waves propagate. However, this time the frequency is not dictated by the spacing of the periodic array but by the frequency of the local oscillator!

We can modify the propagation of wave in the vicinity of a frequency ν by coupling the wave to a local resonance with frequency $\nu_0 \approx \nu$.

Question: Why is the argument that we freed ourselves from length-scale constraints potentially a lie?

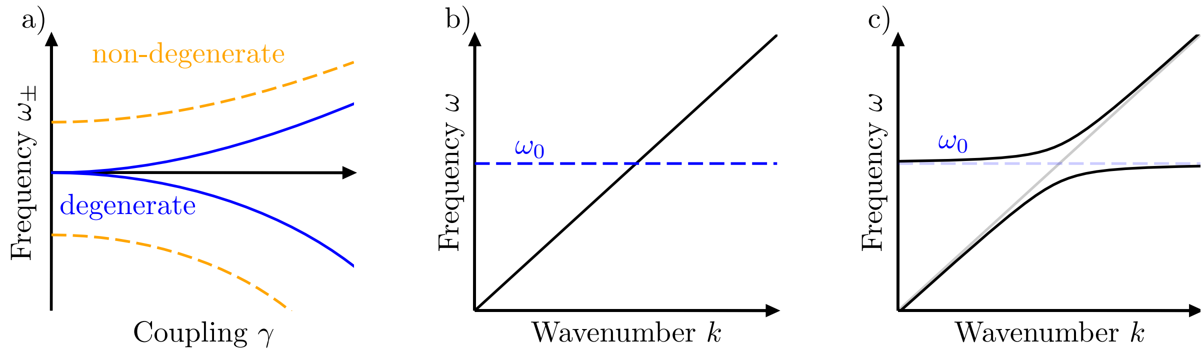


Figure 1.4: a) Effect of the coupling γ on the eigenfrequency of two oscillators. b) Wave dispersion (black) and the location of a local resonance (blue). c) Couling the local resonance to the wave leads to the opening of a band gap at the frequency ω_0 of the local resonance.

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