Chapter 2

Waves in solids

Learning goals

- You know the wave equation.
- You know the concept and differences between the group and the phase velocity.

2.1 The wave equation

To familiarize ourselves with waves and their propagation, we study a simple example of longitudinal waves in thin rods. Let us consider the forces acting on a small section of the rod as shown in Fig. 2.1. The net force on a small segment is given by the difference of the force acting on the left and right cross-section

$$F = -A\sigma(x) + A\sigma(x + \delta x) \tag{2.1}$$

$$= -A[\sigma(x) - \sigma(x) + \delta x \sigma'(x)] = A\delta x \sigma'(x), \tag{2.2}$$

where $\sigma(x)$ is the stress field, or in other words, the force per area A, i.e., the pressure. δx is the infinitesimal thickness of the segment and the prime on σ indicates the derivative with respect to x.

We know that Newton's equations of motion govern the behavior of mechanical systems. Hence

$$ma = F, \tag{2.3}$$

which for our cross-section means

$$ma = \underbrace{A\delta x \rho}_{m} \underbrace{\ddot{u}(x,t)}_{a}, \tag{2.4}$$

where $u(x,t)$ is the displacement field and ρ the density. We therefore find

$$A\delta\rho\ddot{u}(x,t) = A\delta x \sigma'(x,t). \tag{2.5}$$

In order to close this equation, we need a relation between the stress $\sigma(x,t)$ and the displacement $u(x,t)/$ As all segments of a finite rod can take up some of the applied force, the change in length of an elastic object depends on its size:

$$\sigma = E\frac{\Delta L}{L} \qquad \text{or} \qquad \sigma(x,t) = Eu'(x,t) \tag{2.6}$$

with a proportionality factor E called Young's modulus (this is nothing but the "spring constant" k for a continuous system. Note that u' or $\Delta L/L$ is unit-less, so E takes the units of σ, i.e., pressure.). Inserting this relation (2.5) into (2.6) we obtain

$$\boxed{\ddot{u}(x,t) = \frac{E}{\rho}u''(x,t).} \tag{2.7}$$

This is our sought after wave equation. Let us analyze a few properties of this equation.

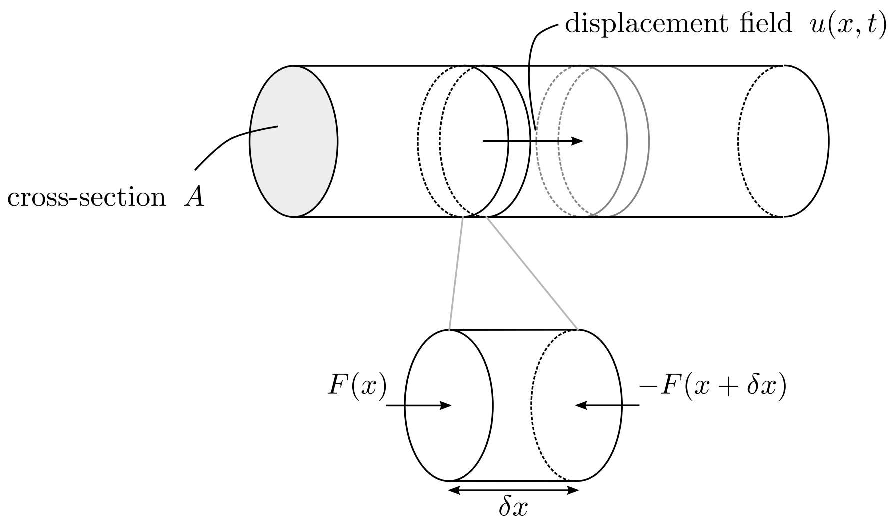


Figure 2.1: Waves in a thin rod.

2.1.1 Traveling wave solutions

Any (twice differentiable) function

$$u(x - vt) \qquad \text{with} \qquad v = \sqrt{\frac{E}{\rho}} \tag{2.8}$$

is a solution of (2.7). To see this, let us insert this ansatz into the wave equation

$$\frac{\rho}{E} v^2 u''(x - vt) = u''(x - vt). \tag{2.9}$$

Therefore, for $v = \sqrt{E/\rho}$, $u(x-vt)$ is indeed as solution. We read off a few interesting properties form this solution

1. Any traveling wave-form is preserved under the evolution of time.
2. The stiffer the material (E larger), the faster the wave.
3. The lighter the material (ρ smaller), the faster the wave.

While these observations are important and useful for the design of metamaterials, we profit from another analysis in terms of "modes".

2.1.2 Eigenmodes of the wave equation

An eigenmode is a natural vibration of the system where all parts oscillate at the same frequency. It is useful t know these modes, as we can construct all solutions from a super-position of such eigenmodes.
For the wave equation these are simply traveling waves

$$u(x,t) = e^{i\varphi(x,t)} = e^{i(kx - \omega t)} \tag{2.10}$$

Inserted into (2.7) we find

$$-c^2 k^2 e^{i(kx-\omega t)} = -\omega^2 e^{i(jx-\omega t)}, \tag{2.11}$$

which means that we have a relation between the wave number k and the angular frequency ω given by

$$\boxed{\omega(k) = c|k| \qquad \text{with} \qquad c = \sqrt{\frac{E}{\rho}}.} \tag{2.12}$$

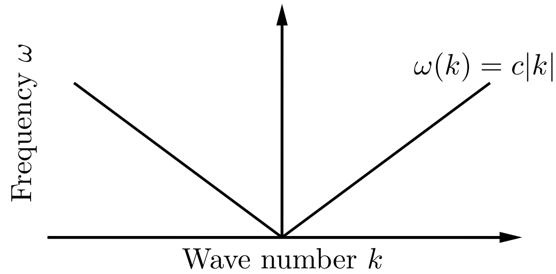


Figure 2.2: Dispersion relation of a simple wave.

This connection between k and ω is called the dispersion relation, see Fig. 2.2. Most of what we are going to do in this lecture is trying to manipulate $\omega(k)$ to achieve our design goals. Let us therefore understand a few more properties of the dispersion relation.

2.1.3 The phase velocity

We have seen in the first lecture that the phase of a wave is a very important property. The point in space of constant phase φ_0 is moving in time

$$0 = \varphi_0(x, t) = kx - \omega(k)t \quad \Rightarrow x_0 = \underbrace{\frac{\omega(k)}{k}}_{v_\varphi} t, \tag{2.13}$$

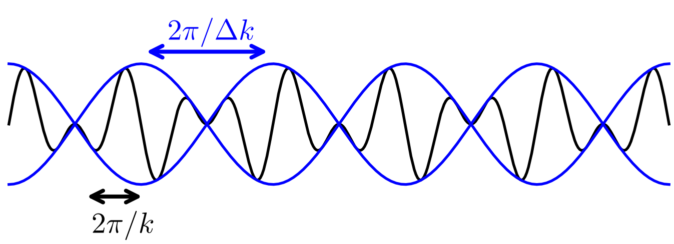
where we defined the phase velocity v_φ.
The phase velocity will become particularly relevant for two and three-dimensional systems, where a non-isotropic $\omega(\mathbf{k})$ (i.e., $\omega(\mathbf{k}) \neq c|\mathbf{k}|$) can lead to distortions of the wave front.

2.1.4 The group velocity

As we are often dealing with wave-packets (like the one depcicted in the beginning where we wrote $u(x - vt)$), the phase velocity is not the only important quantity. Imagine a wave-packet made from modes that are close to some base frequency ω:

$$u(x, t) = e^{i(\omega + \frac{\Delta\omega}{2})t} e^{i(k + \frac{\Delta k}{2})x} + e^{i(\omega - \frac{\Delta\omega}{2})t} e^{i(k - \frac{\Delta k}{2})x} \tag{2.14}$$

$$= 2e^{ikx - i\omega t} \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right). \tag{2.15}$$



The envelope $\cos(\Delta kx/2 - \Delta\omega t/2)$ travels with the velocity $v_g = \Delta\omega/\Delta k$. If we take the limit of $\Delta \to 0$, we find

$$v_g = \frac{\partial\omega(k)}{\partial k} \qquad \text{the group velocity.} \tag{2.16}$$

In the exercises we will see when $v_g = v_\varphi$ and what the significance of the group velocity is!

2.1.5 The wave number k

We have seen that $\exp[ikx - \omega(k)t]$ are solutions to the wave equation (2.7). Note that the wave number k encodes the wavelength $\lambda = 2\pi/k$. Moreover, k controls how the phase of the wave changes when we move in space:

$$\arg[u(x,t_0)] = \arg\left[e^{ikx-\omega(k)t}\right] = kx - \omega(k)t. \tag{2.17}$$

$$\arg[u(x+\Delta x,t_0)] = k(x+\Delta x) - \omega(k)t, \tag{2.18}$$

and therefore

$$\boxed{\Delta\varphi = k\Delta x.} \tag{2.19}$$

And this is the only thing that changes in $u_k(x,t)$ when advancing by Δx. On solutions characterized by "k" we can think that translations act by multiplying by:

$$T_{\Delta x} = e^{i\Delta x k} \tag{2.20}$$

As we will see in the next chapter, this simple property might be lost if our medium is not translational invariant. And this is what we typically do in a metamaterial design: We structure a material in a way that breaks continuous translation-symmetry, e.g., by drilling holes.

2.1.6 A note on waves in solids

In the above example we studied one simple type of elastic waves that occur in solids. OF course, there are many more types of waves like shear-waves, torsional waves, flexural waves, waves that propagate on surfaces like Love or Rayleigh waves. For the purpose of this lecture, however, we focus much more on wave in discrete systems, as for our metamaterial concepts, they are more instructive.

References

1. Landau, L. D. & Lifshitz, E. M. *Theory of Elasticity* (Butterworth-Heinemann, London, 1986).
2. Fetter, A. L. & Walecka, J. D. *Theoretical Mechanics of Particles and Continua* (Dover, 1980).