Chapter 5

Effective negative parameters

Learning goals

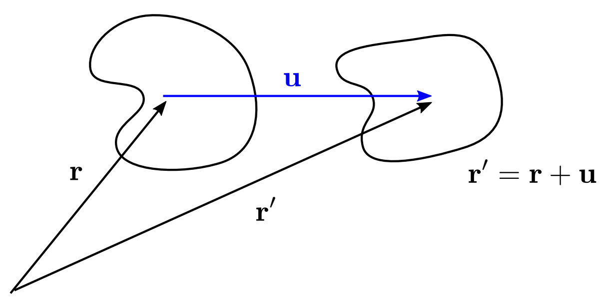
- You can explain why we map our designs to an elastic theory with *effective* parameters.
- You know what strain and stress are and how they are related.
- You know the Helmholtz decomposition.
- You know that doubly negative materials require some thought related to power flow.
- You know Snell's law in doubly negative materials.

So far, we have seen several discrete models of metamaterials to control wave-propagation. In order to better understand what one can achieve with metamaterials, we should connect what we have done so far with the standard literature. For this, however, we need a bit of elasticity theory.

5.1 Elasticity in one hour

5.1.1 The strain tensor

First, we introduce the strain tensor. Imagine a piece of material that we deform:



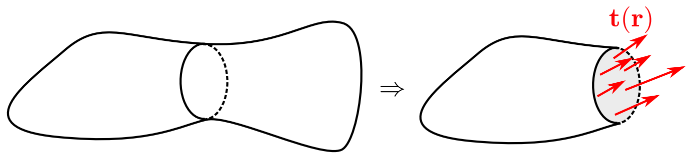
For elastic properties, it is important to know if we *deform* the material. In other words, a constant $\mathbf{u}$ does not lead to any deformations but only translates the whole object. Hence, the quantity of interest is the linear strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{5.1}$$

which captures relative length changes which deform the material.[1] We note that ϵ_{ij} has dimensionless entries. Now we need to connect the strain tensor ϵ_{ij} to forces in the material that try to restore the original shape.

5.1.2 The stress tensor

Forces acting on a body are most easily captured by a traction vector $\mathbf{t}$. Let us make a imaginary cut through our material:



With this we might free forces on the cut that were balanced by the other half. We write for the force $\mathbf{F}$ acting on a surface element dS

$$\mathbf{F} = \mathbf{t} dS. \tag{5.2}$$

Therefore, the traction $\mathbf{t}$ has units of $\mathrm{N/m^2}$ or pressure. The issue with the traction is that it depends on the cut we take. Think of a piece of liquid. If we cut in the xy-plane, pressure will exert on that plane a force in z-direction. Had we cut along the xz-plane, would the pressure give rise to a force in y-direction, and so on. This means, we need again a tensor to capture the relevant physics. The stress tensor τ_{ij} encodes the traction in the three principle axis

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}, \tag{5.3}$$

where $\mathbf{t_x} = (\tau_{xx}, \tau_{yx}, \tau_{zx})$ stands for the traction for a cut normal to x, etc. From this follows immediately that the traction for an arbitrary cut normal to $\hat{\mathbf{n}}$ is given by

$$\mathbf{t}_{\hat{\mathbf{n}}} = \tau \cdot \hat{\mathbf{n}}. \tag{5.4}$$

5.1.3 Hooke's law

We now need a relation between relative deformations and the stress tensor

$$\tau_{ij} = c_{ijkl} \epsilon_{lm}. \tag{5.5}$$

Here, we defined the 4th-order tensor c_{ijkl}. While this looks scary, for our discussion we assume a homogeneous and isotropic medium, where

$$\tau_{ij} = \lambda \delta_{ij} \epsilon_{ll} + 2\mu \epsilon_{ij}. \tag{5.6}$$

[1] Why do we symmetrize? Imagine two close-by points with distance $d\mathbf{r}$. After deformation $d\mathbf{r}' = d\mathbf{r} + \mathbf{u}$. Let us see how the distance changes (using the Einstein summation convention)

$$\begin{aligned} d\mathbf{r}'^2 - d\mathbf{r}^2 &= dx_i' dx_i' - dx_i dx_i = (dx_i + du_i)(dx_i + du_i) - dx_i dx_i = \left(dx_i + \frac{\partial u_i}{\partial x_j} dx_j\right)\left(dx_i + \frac{\partial u_i}{\partial x_l} dx_l\right) - dx_i dx_i \\ &= 2\epsilon_{ij} dx_i dx_i + \mathrm{O}\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2\right]. \end{aligned}$$

In other words, only the symmetric combination in ϵ_{ij} leads to length-changes. It can be shown that the anti-symmetric counter-part corresponds to rigid rotations.

The coefficient λ is called Lamé coefficient and μ is the shear modulus. Note that we made use of the Einstein summation convention where repeated indices are summed over

$$\epsilon_{ll} = \sum_l \epsilon_{ll} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}. \tag{5.7}$$

We often use other names, such as

$$E = \frac{\tau_{xx}}{\epsilon_{xx}}: \qquad \text{Youngs modulus.} \tag{5.8}$$

$$\nu = -\frac{\epsilon_{yy}}{\epsilon_{xx}}: \qquad \text{Poisson ratio.} \tag{5.9}$$

with the relations

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \tag{5.10}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{5.11}$$

$$\mu = \frac{E}{2(1+\nu)}. \tag{5.12}$$

5.1.4 The Poynting vector

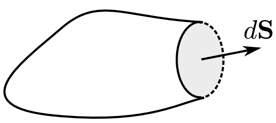


Figure 5.1: Surface element.

Before we move on, it is useful to introduce the power flux in an elastic medium. We know that a force $\mathbf{F}$ acting on a particle with velocity $\mathbf{v}$ delivers a power $\mathbf{F}\cdot\mathbf{v}$ (it has units of $\frac{\mathrm{Nm}}{\mathrm{s}}$). Consider the surface element $d\mathbf{S}$. The force acting on it is given by $\mathbf{t}\cdot dS = \tau\cdot\hat{\mathbf{n}}dS = \tau d\mathbf{S}$. Therefore, the power delivered to the cut is given by

$$P = \mathbf{v}\tau d\mathbf{S}. \tag{5.13}$$

Note that $d\mathbf{S}$ points out of the volume. To get the power delivered *to* a volume, we need to invert the sign and we write for the power flux $\mathbf{J}(\mathbf{r})$

$$\boxed{\mathbf{J}(\mathbf{r}) = -\tau\cdot\dot{\mathbf{u}}.} \tag{5.14}$$

This is nothing but the elastic counter-part to the Poynting vector $\frac{c}{4\pi}\mathbf{E}\wedge\mathbf{B}$ in electromagnetism.

5.2 The elastic wave equation

The Newton's equations of motion are given by

$$\rho\ddot{u}_m = \frac{\partial\tau_{mk}}{\partial x_k}. \tag{5.15}$$

Using

$$\tau_{mk} = \lambda\delta_{mk}\frac{\partial u_j}{\partial x_j} + \mu\left(\frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m}\right) \tag{5.16}$$

we find that

$$\rho\ddot{\mathbf{u}} = (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla\wedge(\nabla\wedge\mathbf{u}). \tag{5.17}$$

We see that things are slightly more complicated than what we did so far. Waves in solids have a complicated vectorial structure (paralleling the tensor nature of stress and strain). But by introducing the longitudinal and transverse potential (the Helmholtz decomposition)[2]

$$\mathbf{u} = \nabla\varphi + \nabla\wedge\boldsymbol{\psi}. \tag{5.18}$$

[2] If we write $\varphi = Ae^{i\mathbf{k}\cdot\mathbf{x}}$ and $\boldsymbol{\psi} = \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{x}}$, we see that $\mathbf{u} = \mathbf{k}A$ and $\mathbf{u} = \mathbf{k}\wedge\mathbf{A}$, respectively. Hence the name longitudinal and transverse potential.

Using this, we arrive at [making use of $\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\wedge(\nabla\wedge\mathbf{u})$]

$$\rho\ddot{\varphi} = (\lambda+2\mu)\nabla^2\varphi, \tag{5.19}$$

$$\rho\ddot{\boldsymbol{\psi}} = \mu\nabla^2\boldsymbol{\psi}. \tag{5.20}$$

By writing $\lambda + 2\mu = E\frac{\nu-1}{(\nu+1)(2\nu-1)} = E^*_{\mathrm{L}}$ and $\mu = E^*_{\mathrm{S}}$ we recover the same expressions for the wave equations as we had before.

5.3 Negative E or ρ

Imagine that for some reason either ρ pr E acquire negative values. While this is impossible for $\omega \to 0$ (masses are intrinsically zero and $E < 0$ would lead to a mechanical collapse), such effective material parameters might arise from metamaterial engineering. This is indeed what we are after here: We try to describe an emergent behavior arising from a (discrete) metamaterial model by reducing it to a simple *effective* elasticity problem, albeit with material parameters E, ν, ρ, etc. that can take values otherwise unattainable.

This approach has the benefit of enabling a simple description in terms of standard elasticity theory, while we can incorporate complicated material designs.

Let us check what happens if either E or ρ take negative values. The wave equation (for longitudinal waves)

$$s_\rho|\rho|\ddot{\varphi} = s_E|E|\nabla^2\varphi. \tag{5.21}$$

Here, $s_\rho = \pm1$ and $s_E = \pm1$ encode the sign of ρ and E, respectively. Assuming

$$\varphi = Ae^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{5.22}$$

we find

$$-s_\rho|\rho|\omega^2 = -s_E|E|(k_x^2 + ky^2 + k_z^2). \tag{5.23}$$

Solving for $k = \sqrt{k_x^2 + ky^2 + k_z^2}$ we obtain

$$k = \sqrt{\frac{s_\rho}{s_E}} \times \sqrt{\frac{|\rho|}{|E|}}\omega. \tag{5.24}$$

We observe that for $s_\rho/s_E = -1$, i.e., if only one of the two parameters is negative, we have a $k \in \mathbb{C}$. In other words we deal with evanescent weaves! We also see, that in the case $s_\rho = s_E = -1$ wave propagation seem to be unaffected by the negativity of E and ρ. We will see, however, that this is not the case if we deal with interfaces between different materials.

5.4 Doubly negative metamaterials

We have seen that for doubly negative materials waves can propagate. Here, we study the effect of double negativity on two examples.

5.4.1 Longitudinal waves in thin rods

We solve for solutions of the form $\mathbf{u} = \nabla\varphi$, i.e., longitudinal waves, in a thin rod shown in Fig. 5.2.

$$\varphi(z) = \begin{cases} \varphi_1^{\mathrm{in}}e^{ik_1z-i\omega t}\varphi_1^{\mathrm{out}}e^{-ik_1z-i\omega t} & z<0, \\ \varphi_2^{\mathrm{out}}e^{ik_2z-i\omega t} & z>0. \end{cases} \tag{5.25}$$

This corresponds to an incoming wave φ_1^{in}, a reflected wave φ_1^{out} and a transmitted wave φ_2^{out}. At the boundary at $z = 0$ we need

$$\mathbf{u}_1(z=0) = \mathbf{u}_2(z=0), \tag{5.26}$$

$$\tau_1(z=0)\cdot\mathbf{z} = \tau_2(z=0)\cdot\mathbf{z}. \tag{5.27}$$

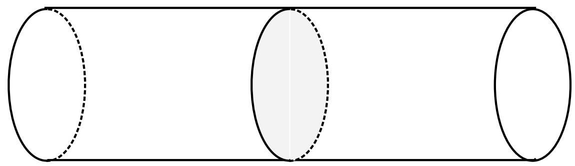


Figure 5.2: A thin rod with a boundary between a regular and a doubly negative metamaterial.

For each segment we have

$$k_i^2 \frac{E_i}{\rho_i} = \omega^2 \qquad \Rightarrow \qquad k_i = \pm\sqrt{\frac{\rho_i}{E_i}}\omega, \tag{5.28}$$

where we have to decide on the sign of k_i. Let us first match the boundary conditions

$$u_1^z(z=0) = ik_1\varphi_1^{\text{in}} e^{-i\omega t} - ik_1\varphi_1^{\text{out}} e^{-i\omega t}, \tag{5.29}$$

$$u_2^z(z=0) = ik_2\varphi_2^{\text{out}} e^{-i\omega t}. \tag{5.30}$$

For the stress tensors, we need

$$\left.\frac{\partial u_1^z}{\partial z}\right|_{z=0} = -k_1^2\varphi_1^{\text{in}} e^{-i\omega t} - k_1^2\varphi_1^{\text{out}} e^{-i\omega t}, \tag{5.31}$$

$$\left.\frac{\partial u_2^z}{\partial z}\right|_{z=0} = -k_2^2\varphi_2^{\text{out}} e^{-i\omega t}. \tag{5.32}$$

From this, we obtain

$$k_1(\varphi_1^{\text{in}} - \varphi_1^{\text{out}}) = k_2\varphi_2^{\text{out}}, \tag{5.33}$$

$$E_1 k_1^2(\varphi_1^{\text{in}} + \varphi_1^{\text{out}}) = E_2 k_2^2 \varphi_2^{\text{out}}. \tag{5.34}$$

We set $\varphi_1^{\text{in}} = 1$ and solve for $\varphi_{1/2}^{\text{out}}$:

$$\varphi_1^{\text{out}} = -\frac{E_1k_1 - E_2k_2}{E_1k_1 + E_2k_2}, \qquad \varphi_2^{\text{out}} = \frac{2E_1k_1^2}{k_2(E_1k_1 + E_2k_2)}. \tag{5.35}$$

Now is a good moment to take care of the signs. We want φ_1^{in} to be an incoming wave. Let us calculate the Poynting vector for this wave

$$\mathbf{J}^{\text{in},1} = -\tau \cdot \dot{\mathbf{u}} = E_1\omega k_1^3(\varphi_1^{\text{in}})^2\hat{\mathbf{z}}. \tag{5.36}$$

$\Rightarrow$ for $k_1 = +\sqrt{\rho_1/E_1}$ we have indeed an incoming wave described by φ_1^{in}. Going through the same calculation for φ_2^{out} we find

$$\mathbf{J}^{\text{out},2} = E_2\omega k_2^3(\varphi_2^{\text{out}})^2\hat{\mathbf{z}}. \tag{5.37}$$

For this to be an *out-going* wave we need

$$k_2 = -\sqrt{\frac{\rho_2}{E_2}} = \text{sign}(E_2)\sqrt{\frac{\rho_2}{E_2}}. \tag{5.38}$$

Let us introduce a few helpful quantities. First the elastic impedance

$$z_\alpha = \sqrt{\rho_\alpha E_\alpha}, \tag{5.39}$$

and the elastic index of refraction

$$n_\alpha = \mathrm{sign}(E_\alpha)\sqrt{\frac{\rho_\alpha}{E_\alpha}}. \tag{5.40}$$

Using these definitions we can summarize this solution by writing

$$\frac{k_1}{k_2} = \frac{n_1}{n_2}, \tag{5.41}$$

$$\varphi_1^{\mathrm{out}} = \frac{z_2 - z_1}{z_1 + z_2} = -\frac{1 - \frac{z_2}{z_1}}{1 + \frac{z_2}{z_1}}, \tag{5.42}$$

$$\varphi_2^{\mathrm{out}} = \frac{n_1}{n_2}\frac{2}{1 + \frac{z_2}{z_1}}. \tag{5.43}$$

What do we learn from this exercise?

- If we match impedances, i.e., $z_1 = z_2$, there is no reflected wave.
- For $E < 0 \;\Rightarrow\; n < 0$ and the sign of k is inverted to have a causal energy flow.

5.4.2 The Veselago lens

We finish this chapter with an application of a doubly negative material in two dimensions. Imagine the following situation depicted in Fig. 5.3.

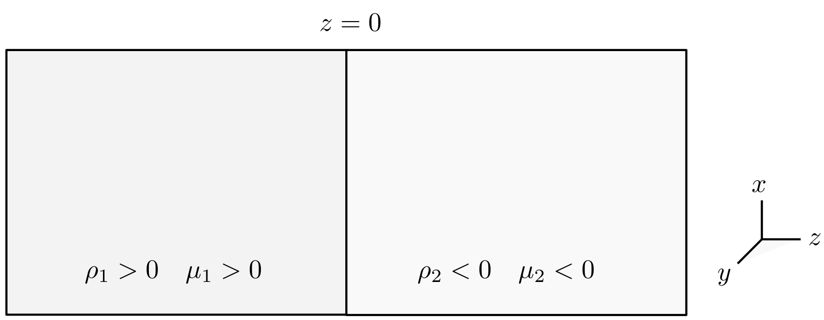


Figure 5.3: Boundary between two materials for a Veselago lens.

We consider shear waves of the form

$$\mathbf{u} = \nabla \wedge \boldsymbol{\psi} \qquad \text{with} \qquad \boldsymbol{\psi} = \begin{pmatrix} A_x \\ 0 \\ A_z \end{pmatrix} e^{i(k_x,0,k_z)\cdot\mathbf{x} - i\omega t}, \tag{5.44}$$

which means

$$\mathbf{u} = (A_x i k_z + A_z i k_x) e^{i(k_x,0,k_z)\cdot\mathbf{x} - i\omega t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{5.45}$$

or in other words shear waves traveling in $(k_x, 0, k_z)$ direction with a deformation in the $\hat{\mathbf{y}}$-direction. Again, the wave equation dictates

$$\rho_i \ddot{\boldsymbol{\psi}} = \mu_i \nabla^2 \boldsymbol{\psi} \qquad \text{or} \qquad \rho_i \omega^2 = \mu_i (k_x^2 + k_y^2 + k_z^2). \tag{5.46}$$

Let us again assume an incoming wave form the left and a transmitted and reflected wave

$$\boldsymbol{\psi}(x,z) = \begin{cases} \boldsymbol{\psi}_1(x,z) = \begin{pmatrix} A_x^{\mathrm{in}} \\ 0 \\ A_z^{\mathrm{in}} \end{pmatrix} e^{ik_z^1 z + ik_x^1 x - i\omega t} + \begin{pmatrix} A_x^{\mathrm{out}} \\ 0 \\ A_z^{\mathrm{out}} \end{pmatrix} e^{-ik_z^1 z + ik_x^1 x - i\omega t} & z < 0, \\ \boldsymbol{\psi}_2(x,z) = \begin{pmatrix} B_x^{\mathrm{out}} \\ 0 \\ B_z^{\mathrm{out}} \end{pmatrix} e^{ik_z^2 z + ik_x^2 x - i\omega t} & z > 0. \end{cases} \tag{5.47}$$

The boundary conditions are given by

$$\mathbf{u}_1(z=0) = \mathbf{u}_2(z=0), \tag{5.48}$$
$$\tau_1(z=0)\cdot\hat{\mathbf{z}} = \tau_2(z=0)\cdot\hat{\mathbf{z}}. \tag{5.49}$$

For the first equation we have to assure that

$$(A_x^{\mathrm{in}} ik_z^1 - A_z^{\mathrm{in}} ik_x^1)e^{ik_x^1 x} + (-A_x^{\mathrm{out}} ik_z^1 - A_z^{\mathrm{out}} ik_x)e^{ik_x^1 x} \overset{!}{=} (B_x^{\mathrm{out}} ik_z^2 - B_z^{\mathrm{out}} ik_x^2)e^{ik_x^2 x}. \tag{5.50}$$

If this shall hold for all x we need

$$k_x^1 = k_x^2. \tag{5.51}$$

This was to be expected as we do not break translational symmetry in x-direction. Moreover, we certainly need that all waves have the same frequency ω. That means that the above equation together with

$$\rho_i\omega^2 = \mu_i[(k_x^i)^2 + (k_z^i)^2] \tag{5.52}$$

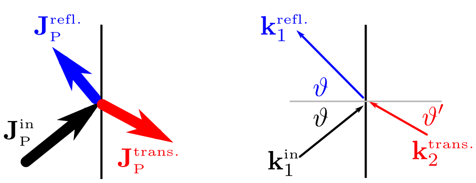


Figure 5.4: Snell's law.

fixes $|k_z|^2$. To determine the sign of k_z we go through the Poynting argument again. It is easy to see that a negative $\mu_2 < 0$ again reverses the sign of the power flux with respect to $\mu_1 > 0$. We therefore again have to take $\mathrm{sign}(k_z^2) = -1$. From these considerations we can determine Snell's law

$$\sin(\vartheta) = \frac{k_x^1}{\sqrt{(k_x^1)^2 + (k_z^1)^2}} \tag{5.53}$$
$$\sin(\vartheta') = -\frac{k_x^2}{\sqrt{(k_x^2)^2 + (k_z^2)^2}} \tag{5.54}$$

Hence, we have

$$\boxed{\frac{\sin(\vartheta)}{\sin(\vartheta')} = \frac{\mathrm{sign}(\mu_2)\sqrt{\rho_2/\mu_2}}{\mathrm{sign}(\mu_1)\sqrt{\rho_1/\mu_1}} = \frac{n_2}{n_1}.} \tag{5.55}$$

For the full scattering solution we need to solve the compatibility conditions above. However, here we are not interested in the amount of power transferred, but only in the direction of in and out-going waves. Veselago realized that [1] (in the context of electromagnetic waves) the above Snell's law for doubly negative materials leads to a perfect flat lens. He considered a situation shown in Fig. 5.5. We see that this sandwich gives rise to a perfect lens.

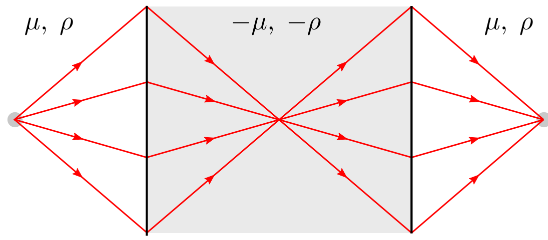


Figure 5.5: A Veselago lens.

5.4.3 Superlensing

Pendry realized in 2000 that a slab of a doubly negative material not only acts as a planar lens but can also enhance features that are normally suppressed due to the diffraction limit.
The diffraction limit arises from the following consideration. We know that

$$\omega^2/c^2 = k_x^2 + k_y^2 + k_z^2. \tag{5.56}$$

Let us assume that we have an object we want to image in the xy-plane at $z = 0$ of the form $f(x, y) = \Theta(|x| - a/2)$. The Fourier transform of this object is given by

$$\hat{f}(k_x, k_y) = \int dx\, e^{ik_x x}\Theta(|x| - a/2) \int dy\, e^{ik_y y} = \tag{5.57}$$

$$= \langle \frac{1}{ik_x} e^{ik_x x}\Big|_{x=-a/2}^{a/2} \rangle 2\pi\delta(k_y) \tag{5.58}$$

$$= 4\pi \frac{\sin(k_x a/2)}{k_x}\delta(k_y). \tag{5.59}$$

We see that to reproduce $f(x, y)$ we need k_x to be arbitrarily large. However, we can write for

$$k_z(k_x, k_y, \omega) = \pm\sqrt{(\omega/c)^2 - k_x^2 - k_y^2}. \tag{5.60}$$

We immediately see that for $k_x^2 + k_y^2 > (\omega/c)^2$, k_z will become purely imaginary and features small than c/ω will not propagate!
For the example above, we get the acoustic image at a distance d by calculating

$$f_d(x, y) = \frac{1}{(2\pi)^2} \int dk_x dk_y \hat{f}(k_x, k_y) e^{ik_z(k_x,k_y,\omega)d} e^{-i(k_x x + k_y y)}. \tag{5.61}$$

We see that Fourier components with a large wave number are damped way faster, as these components turn into evanescent waves at the frequency we probe the system.
We investigate how this changes if we add again a slab of a doubly negative material. For simplicity we look at the evanescent waves, where $k_x^2 > \frac{\rho}{\mu}\omega^2$. Moreover, we constrain ourselves to the situation where we have
We first concentrate on the left boundary and write for the potential

$$\boldsymbol{\psi} = \begin{cases} \boldsymbol{\psi}_1(x, z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{ik_z z + ik_x x - i\omega t} + \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} e^{-ik_z z + ik_x x - \omega t} & z < 0, \\ \boldsymbol{\psi}_2(x, z) = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} e^{ik'_z z + ik_x x - \omega t} & z > 0. \end{cases} \tag{5.62}$$

Clearly the wave vectors have to fulfill

$$k_z = +i\sqrt{k_x^2 - \omega^2}, \tag{5.63}$$

$$k_z' = +i\sqrt{k_x^2 - \omega^2/|\mu|}. \tag{5.64}$$

How did we choose those signs? As we are dealing with evanescent waves, $\mathbf{J}^z = 0$ and no power-flux argument can be invoked. However, we consider a decaying field from the left and therefore the field on the right $z > 0$ should also decay in the positive z-direction to preserve causality. We can now match the boundary conditions

$$u_1(z=0) = u_2(z=0), \tag{5.65}$$

$$\tau_1(z=0)\cdot\hat{\mathbf{z}} = \tau_2(z=0)\cdot\hat{\mathbf{z}}. \tag{5.66}$$

The first line is easy

$$k_z - rk_z = tk_z'. \tag{5.67}$$

For the second we need to calculate τ from $\mathbf{u}$. We only need the three components $\tau_{\alpha z}$ for $\alpha = x, y, z$ as we match $\tau\cdot\hat{\mathbf{z}}$. Moreover, we only have u^y which depends on x and z. Therefore $\epsilon_{ll} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0$. This leaves us with

$$\tau_{\alpha z} = \underbrace{\lambda\delta_{\alpha z}\epsilon_{ll}}_{=0} + 2\mu\epsilon_{\alpha z} = 2\mu\frac{1}{2}\left(\frac{\partial u_\alpha}{\partial z} + \underbrace{\frac{\partial u_z}{\partial x_\alpha}}_{=0}\right) = \mu\frac{\partial u_y}{\partial z} = \begin{cases} -k_z^2(1+r)e^{ik_z z + ik_x x - i\omega t} & z < 0, \\ -k_z'^2 t e^{ik_z' z + ik_x x - i\omega t} & z > 0. \end{cases}$$

And with this we need

$$k_z^2(1+r) = \mu k_z'^2 t. \tag{5.68}$$

Solving for t and r we find

$$t = 2\frac{k_z^2}{k_z'}\frac{1}{k_z + \mu k_z'}; \qquad r = \frac{\mu k_z' - k_z}{k_z + \mu k_z'}. \tag{5.69}$$

Note that for $\mu \to -1$ we have $k_z' \to k_z$ and both r and t diverge! In particular $t^2 + r^2$ diverges as well. This is only possible because we deal with evanescent waves that carry no power-flux! If we now try to move towards the description of a finite slab with thickness d, we also need the same expressions for the right boundary

$$k_z'(1-r) = tk_z, \tag{5.70}$$

$$\mu k_z'^2(1+r) = tk_z^2. \tag{5.71}$$

From which we obtain

$$t' = 2\frac{k_z'^2}{k_z}\frac{\mu}{k_z + \mu k_z'}; \qquad r' = \frac{k_z - \mu k_z'}{k_z + \mu k_z'}. \tag{5.72}$$

Naively, one could expect the transmission function to be

$$T_{\text{naive}}(d) = te^{ik_z' d}t', \tag{5.73}$$

where the first t is the left boundary, the exponential describes the transmission inside the slab and t' encodes the effects of the second boundary. However, the reflected wave inside the slab hits the left boundary again, etc. So the full transmission is given by

$$T(d) = tt'e^{ik_z'd} + tt'r'^2e^{3ik_z'd} + tt'r'^4e^{5ik_z'd} + tt'r'^6e^{7ik_z'd} + \dots \tag{5.74}$$

$$= tt'e^{ik_z'd}\left[1 + r'^2e^{2ik_z'd} + r'^4e^{4ik_z'd} + \dots\right] = \frac{tt'e^{ik_z'd}}{1 - r'^2e^{2ik_z'd}}. \tag{5.75}$$

Let us analyze this expression. $k'_z = i\sqrt{k_x^2 - \omega^2/|\mu|}$ with $k_x^2 > \omega^2/|\mu| \Rightarrow$ both exponential factors are much smaller than one for large enough d. If r'^2 and tt' would be well behaved, we could neglect $r'^2 \exp[-2ik'_z d]$ with respect to 1 and we would obtain the naive result. However, for $\mu \to -1$, we need to be more careful

$$\lim_{\mu\to-1} T(d) = \lim_{\mu\to-1} 4k_z k'_z \frac{\mu^2}{(k_z + \mu k'_z)^2} \frac{e^{ik'_z d}}{1 - \left(\frac{k_z - \mu k'_z}{k_z + \mu k'_z}\right)^2 e^{-2ik'_z d}} \tag{5.76}$$

$$= \lim_{\mu\to-1} 4k_z k'_z e^{ik'_z d} \frac{1}{(k_z + \mu k'_z)^2 - (k_z - \mu k'_z)^2 e^{-2ik'_z d}} \tag{5.77}$$

$$= 4k_z^2 e^{ik_z d} \frac{1}{(k_z - k'_z)^2 - (k_z + k'_z)^2 e^{-2ik'_z d}} \overset{k_z \to k'_z}{=} e^{-ik_z d}. \tag{5.78}$$

We found an astonishing result: Evanescent waves are exponentially enhanced while passing through the doubly negative material! This famous results by Pendry [2] established the concept of a superlens built from metamaterials.

References

1. Veselago, V. G. "The electrodynamics of substances with simultaneously negative values of ϵ and μ". *Sov. Phys. Usp.* **10,** 509 (1968).
2. Pendry, J. B. "Negative Refraction Makes a Perfect Lens". *Phys. Rev. Lett.* **85,** 3966 (2000).