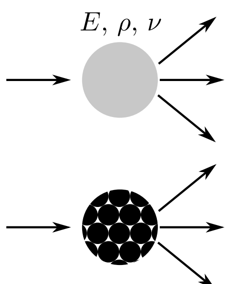
Chapter 6

The continuum limit

Learning goals

- You know how to define effective parameters for a discrete system.
- You know how to find the effective mass for a simple setup.
- You know how to determine the effective spring constant.

In the chapters 3 and 4 we have seen how we can shape the propagation of waves by using periodic structures or by employing local resonances. In the last chapter, on the other hand, we have seen how one can obtain interesting phenomena within the simple framework of a wave-equation, albeit with "unnatural" parameters $E < 0$ and $\rho < 0$.
How to map a structured metamaterial to a simple wave equation with effective parameters is a highly non-trivial task. We can group approaches to do so into two classes. First, if we deal with a continuous medium to which we add resonances of periodic array of scatterers one can do the following



- Solve how a circular object with E, ρ, ν scatters a wave.
- Solve the problem of inclusions in a circular metamaterial and calculate how it scatters an incoming wave.
- By comparing the two results, fix the effective E, ρ, ν of this circular material.

This program is called homogenization. Here, we want to follow a different path. We would like to know how to map a discrete model with an interesting band structure to an effective "continuum" theory. In this case we introduce "natural" coordinates ξ_n and think of the blue bodies in Fig. 6.1 as rigid, despite the fact that they might have an interesting inner life. The effective mass m_{eff} and the effective spring constant are then defined via

$$m_{\text{eff}} = \frac{F_n}{\partial_t^2 \xi_n} \qquad \text{and} \qquad f_{\text{eff}}^{nm} = \frac{F_{nm}}{\xi_n - \xi_m}, \tag{6.1}$$

where F_n are the (external) forces on block n and F_{nm} the forces between n and m.

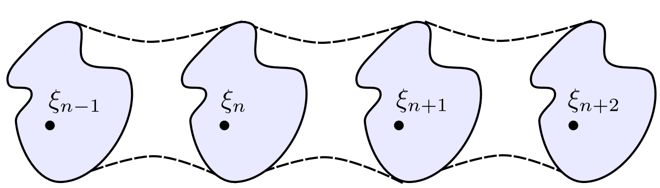


Figure 6.1: Effective coordinates of a more complicated body.

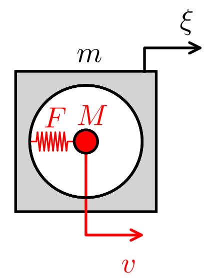


Figure 6.2: Mass in mass system.

6.1 Mass in mass

We start by revisiting the mass-in-mass system, cf. Fig. 6.2. Imagine that we would not see the interior mass M. So we should find an effective description for the "outer" mass only. Let us write

$$M\ddot{v} = F(\xi - v). \tag{6.2}$$

We assume harmonic motion

$$v = v_0 e^{i\omega t}, \tag{6.3}$$
$$\xi = \xi_0 e^{i\omega t}. \tag{6.4}$$

Inserted into (6.2) we get

$$-\omega^2 M v_0 = F(\xi_0 - v_0) \tag{6.5}$$

and therefore

$$v_0 = \frac{F\xi_0}{F - \omega^2 M}. \tag{6.6}$$

By writing $\omega_M = \sqrt{F/M}$, we get

$$v_0 = \frac{\omega_M^2}{\omega_M^2 - \omega^2}\xi_0. \tag{6.7}$$

Let us now introduce the effective mass. We know that

$$\frac{dP}{dt} = \partial_t(m\partial_t\xi + M\partial_t v) = F_{\text{ext}}. \tag{6.8}$$

As we do not "know" about the mass M we would write

$$m_{\text{eff}}\partial_t^2\xi = F_{ext}. \tag{6.9}$$

As Eq. (6.7) is independent of time, we have

$$\partial_t v = \frac{\omega_M^2}{\omega_M^2 - \omega^2}\partial_t\xi. \tag{6.10}$$

Using this in Eq. (6.8), we find

$$\partial_t\left(m\partial_t\xi + M\frac{\omega_M^2}{\omega_M^2 - \omega^2}\partial_t\xi\right) = F_{\text{ext}} \tag{6.11}$$
$$\partial_t\left[m\left(1 + \frac{M}{m}\frac{\omega_M^2}{\omega_M^2 - \omega^2}\right)\partial_t\xi\right] = F_{\text{ext}} \tag{6.12}$$
$$m_{\text{eff}}\partial_t^2\xi = F_{\text{ext}}. \tag{6.13}$$

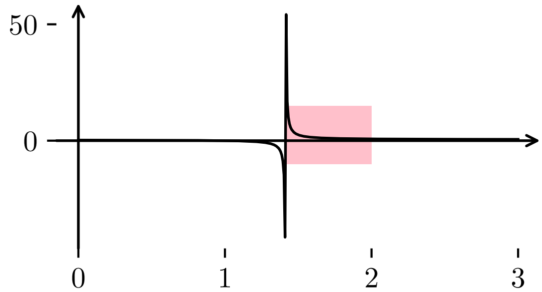


Figure 6.3: Effective mass.

In the last line we defined

$$m_{\text{eff}} = m\left(1 + \frac{M}{m}\frac{\omega_M^2}{\omega_M^2 - \omega^2}\right). \tag{6.14}$$

We observe:

- for $m_{\text{eff}} < 0$: You push the mass and it moves in your direction.
- m_{eff} is strongly frequency dependent!
- $\omega \to 0$: $m_{\text{eff}} = m + M$.

Let us now use this effective mass description to solve the chain of such local resonators

$$m_{\text{eff}}(\omega)\ddot{\xi}_n = -f(2\xi_n - \xi_{n+1} - \xi_{n-1}). \tag{6.15}$$

As usual we assume $\xi_n = \exp ikna$ to obtain

$$m_{\text{eff}}(\omega)\omega^2 = 2f[1 - \cos(ka)] = 4f\sin^2(ka/2). \tag{6.16}$$

When we solve for k we find

$$k = \frac{2}{a}\arcsin\left[\frac{1}{2}\sqrt{\frac{m_{\text{eff}}(\omega)}{f}}\omega\right]. \tag{6.17}$$

From this expression we see that k becomes imaginary if

1. $m_{\text{eff}} < 0 \Leftarrow$ Band gap
2. $\frac{1}{2}\sqrt{m_{\text{eff}}/f}\,\omega > 1 \Leftarrow$ above band edge

6.2 Effective spring

We have seen that a “hidden” mass can give rise to an effective negative mass. What about negative springs? Let us consider the system shown in Fig. 6.4. The equations of motion are

$$m\ddot{y} = F_2, \tag{6.18}$$
$$F_2 = 2(F_1 - 2\kappa_2\bar{x})\tan(\alpha), \tag{6.19}$$
$$F_1 = \kappa_1(x - \bar{x}). \tag{6.20}$$

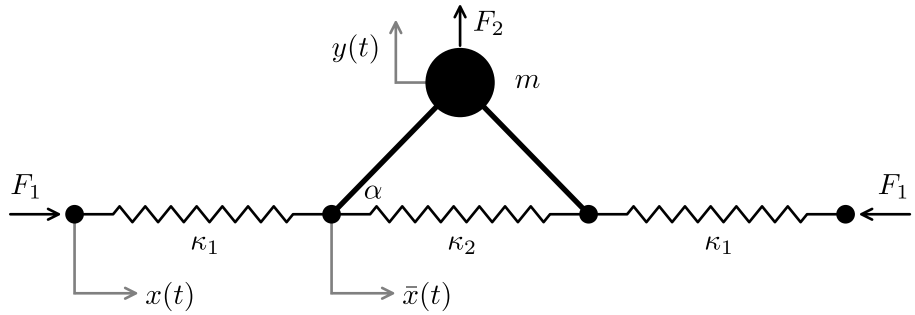


Figure 6.4: Effective sping.

From the last line we infer that $\bar{x} = x - F_1/\kappa_1$. Inserted into the middle line we get $F_2 = 2[F_1 - 2\kappa_2(x - F_1/\kappa_1)]\tan(\alpha)$. Moreover, we can use that $\bar{x} = \tan(\alpha)y$ which yields $y = (x - F_1/\kappa_1)/\tan(\alpha)$. Using all of this in the first line we find

$$-m\omega^2\left(1 - \frac{F_1}{\kappa_1}\right) = 2\left[F_1 - 2\kappa_2\left(x - \frac{F_1}{\kappa_1}\right)\right]\tan^2(\alpha) \tag{6.21}$$

$$\Rightarrow x = F_1\frac{m\omega^2 - 2(\kappa_1 + 2\kappa_2)\tan^2(\alpha)}{\kappa_1[m\omega^2 - 4\kappa_2\tan^2(\alpha)]}. \tag{6.22}$$

We can now define

$$x = \frac{F_1}{2\kappa_{\text{eff}}} \tag{6.23}$$

and write

$$\kappa_{\text{eff}} = \frac{1}{2}\left[\frac{(\omega^2 - \frac{4\kappa_2}{m}\tan^2(\alpha)\kappa_1)}{\omega^2 - \omega_0^2}\right], \tag{6.24}$$

with $\omega_0^2 = \frac{2(\kappa_1+2\kappa_2)}{m}\tan^2(\alpha)$. We check for the sanity of this results by taking

$$\lim_{\omega\to 0}\frac{1}{\kappa_{\text{eff}}} = \frac{1}{\kappa_2} + \frac{2}{\kappa_1}, \tag{6.25}$$

which is what we should obtain! Analogous to the effective mass, the effective spring constant can be negative, cf. Fig. 6.5.
Again, we can bunch such elements together to find the equations of motion for a chain made from effective springs

$$m\ddot{\xi}_n = -\kappa_{\text{eff}}(2\xi_n - \xi_{n-1} - \xi_{n+1}) \tag{6.26}$$

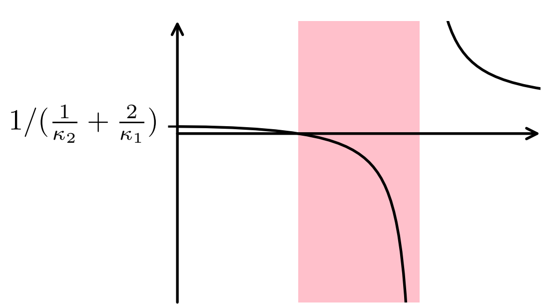


Figure 6.5: Effective spring constant.

Not so surprisingly the wave number

$$k = \frac{2}{a}\text{arcsin}\left(\sqrt{\frac{m}{\kappa_{\text{eff}}(\omega)}}\omega\right) \tag{6.27}$$

is again complex in the region where κ_{eff} is negative and above the upper band edge.

6.3 Double negativity

We now combine the two ingredients, an effective mass and an effective spring as shown in Fig. . We immediately find

$$m_{\text{eff}}\omega^2 = 4\kappa_{\text{eff}}\sin^2(ka/2). \tag{6.28}$$

Task: Write m_{eff} and κ_{eff} in a way that transparently captures the negative sections. Then solve the above equations for ω and plot the solutions as a function of k and vary the tuning parameters. Convince yourself that indeed for $m_{\text{eff}}\kappa_{\text{eff}} < 0$ you and up in a band gap and for $m_{\text{eff}} < 0$ and $\kappa_{\text{eff}} < 0$ you have indeed $\partial_k\omega(k) < 0$.

References

1. Zhou, X., Liu, X. & Hu, G. "Elastic metamaterials with local resonances: an overview". *Theor. Appl. Mech. Lett.* **2,** 041001 (2012).
2. Lee, S. H. & Wright, O. B. "Origin of negative density and modulus in acoustic metamaterials". *Phys. Rev. B* **93,** 024302 (2016).