Exercise 1. ***Energy transport in a one-dimensional crystal***

Consider the one-dimensional harmonic crystal depicted in Fig. 1. The equation of motion for the n-th site is given by

$$m\frac{d^2u_n}{dt^2} = \gamma(u_{n+1} - u_n) - \gamma(u_n - u_{n-1})\,, \tag{1}$$

where u_n is the displacement of the n-th site along the chain. The coefficients m and γ denote the mass of a single site and the spring constant of the coupling springs likewise.

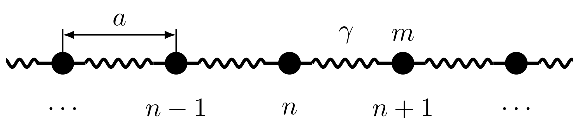

Figure 1: One-dimensional crystal with lattice constant a.

(a) Find the dispersion relation $\omega(k)$ of the crystal by solving Eq. (1). Use the ansatz of a traveling wave

$$u_n = Ae^{ikna}e^{i\omega t}\,. \tag{2}$$

(b) Calculate the phase veloctiy

$$v_\varphi = \frac{\omega}{k}\,, \tag{3}$$

and the group velocity

$$v_g = \frac{d\omega}{dk}\,. \tag{4}$$

For what type of dispersion relations are they equal?

(c) Assume now that there is a single traveling wave [Eq. (2)] propagating through the system. Calculate the time-averaged energy per unit cell.

Hint: For complex-valued solutions, the time-averaged energy of a harmonic oscillator can be written as

$$\langle E\rangle \propto \frac{1}{2}\dot{u}\dot{u}^* + \frac{1}{2}\omega^2 uu^*\,, \tag{5}$$

where * denotes complex conjugation. The first term is the kinetic energy and the second one is due to the potential energy.

(d) Each spring is performing work on the mass points. Calculate the associated time-averaged power per unit cell.

Hint: The time-averaged power can be conveniently calculated as the real part of the complex valued power $\langle P\rangle = \mathrm{Re}\{Fv\}$, where F and v denote a complex valued force and velocity likewise.

(e) By relating the average energy density and the power per unit cell, find at which velocity energy is transported through the structure. How does it relate to the phase and group velocities?