Chapter 4

Vortices

Learning goals

- How does the velocity in a BEC depend on the phase of the wave function?
- Why is the circulation in a condensate quantized?
- What is the azimuthal velocity in presence of a vortex?
- What is the density profile of a vortex?

4.1 Overview

The Gross-Pitaevskii equation which we derived earlier is not only useful to find the ground state wave function of a condensate, but is also capturing the excitations of a BEC. One particular excitation, which is a special property of a superfluid, are vortices. They are often called as experimental "smoking gun" for the existence of superfluidity. While they could be observed in superfluid Helium only indirectly, they form an important research area in the field of dilute quantum gases. In this chapter, we will use the time-dependent Gross-Pitaevskii equation to derive a condition for the quantized circulation in a condensate under rotation. This leads to a non-intuitive velocity field which is strongly different from the velocity field we know from normal rotating liquids. The GPE helps us to find the density structure of a vortex, and the energy related to the existence of vortices, as well as a critical angular velocity for the creation of vortices.

4.2 Continuity equation and irrotational flow

Our starting point is the time-dependent Gross-Pitaevskii equation, which can be derived again with a variational approach as discussed in the previous chapter for the time-independent case,

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t) + U_0|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t)\,. \tag{4.1}$$

To be consistent with the formulation for the time-independent GPE, we require that the wave function $\psi(\mathbf{r},t)$ must develop under stationary conditions according to $e^{-i\mu t/\hbar}$. Knowing that the chemical potential corresponds to the energy needed to add one particle to the system, this phase evolution can be microscopically understood as the matrix element of the annihilation operator $\hat{\psi}(\mathbf{r})$ changing the system from a state with N to $N-1$ particles. The state with N particles will evolve as $e^{-iE_N t/\hbar}$, while the state with $N-1$ particles evolves like $e^{-iE_{N-1}t/\hbar}$, such that

$$\psi(\mathbf{r},t) = \langle N-1|\hat{\psi}(\mathbf{r})|N\rangle \propto e^{-i(E_N - E_{N-1})t/\hbar}\,. \tag{4.2}$$

For large N, this energy difference can be replaced by the derivative of the energy with respect to the number of particles, which is the chemical potential.
From the time-dependent Gross-Pitaevskii equation, we can derive the continuity equation. This will show us how the velocity field of the condensate is connected to the wave function. We multiply the time-dependent GPE with ψ^*, and subtract the complex conjugate [1],

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\hbar}{2mi}\boldsymbol{\nabla}(\psi^*\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\psi^*) = 0\,. \tag{4.3}$$

This equation can be compared with the standard continuity equation usually derived from the linear Schrödinger equation

$$\frac{\partial}{\partial t}n + \boldsymbol{\nabla}(n\mathbf{v}) = 0\,, \tag{4.4}$$

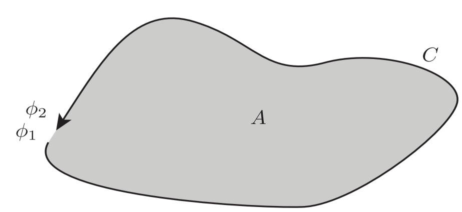
where the particle density $n = |\psi|^2$. We thus identify the velocity field of the condensate as

$$\mathbf{v} = \frac{\hbar}{2mi}\frac{\psi^*\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\psi^*}{|\psi|^2}\,. \tag{4.5}$$

The wave function of the condensate must be single valued, and we use the Ansatz $\psi = fe^{i\phi}$, which contains the modulus f and the phase ϕ of the wave function. The density is then given by $n = |\psi|^2 = f^2$. Plugging this Ansatz into the expression for the velocity, we get[2]

$$\mathbf{v} = \frac{\hbar}{2mi}\frac{\psi^*\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\psi^*}{|\psi|^2} = \frac{\hbar}{m}\boldsymbol{\nabla}\phi\,. \tag{4.6}$$

This is an important result: the gradient of the phase of the wave function describes the velocity field of the condensate. The expression $\frac{\hbar}{m}\phi$ can thus be considered as "velocity potential". This condition actually heavily constricts the possible motion of a condensate as we will see below.



It is useful to introduce the concept of circulation $\Gamma = \oint_C \mathbf{v}\mathbf{dl}$ around a closed contour C enclosing the area A. A consequence of the definition of the velocity as a conservative vector field is that the circulation must vanish for a non-singular phase ϕ, which means that the motion of the condensate must be irrotational,

$$\Gamma = \oint_C \mathbf{v}\mathbf{dl} = \frac{\hbar}{m}\oint_C \boldsymbol{\nabla}\boldsymbol{v}\mathbf{dl} = \frac{\hbar}{m}\int_A \boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = 0\,. \tag{4.7}$$

It seems as if a superfluid could not be put into rotation. If you think of a slowly rotating bucket filled with a normal fluid, then the fluid will rotate together with the fluid just as a rigid body. In contrast, with the above argument for the circulation a superfluid seems to stay at rest in a rotating container. However, one can still fulfill irrotationality and the existence of non-zero angular momentum if the phase ϕ has a singularity!
If we allow for such a singularity of the phase, the wave function still must be single valued. It follows that the the change $\Delta\phi = \phi_2 - \phi_1$ of the phase of the wave function, when going around the closed contour C must be a multiple of 2π, such that the circulation actually becomes quantized according to

$$\Gamma = \oint \mathbf{v}\mathbf{dl} = \frac{\hbar}{m}\oint \boldsymbol{\nabla}\phi\mathbf{dl} = \frac{\hbar}{m}(\phi_2 - \phi_1) = 2\pi l\frac{\hbar}{m} = l\frac{h}{m}\,, \tag{4.8}$$

[1] using $\boldsymbol{\nabla}^2\psi\psi^* = \boldsymbol{\nabla}(\psi^*\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\psi^*)$
[2] using $\boldsymbol{\nabla}(fe^{i\phi}) = e^{i\phi}\boldsymbol{\nabla}f + fi\boldsymbol{\nabla}\phi e^{i\phi}$

where l is an integer describing the winding number of the singularity.
We have shown that if we allow for a singularity in the wave function, the superfluid can indeed carry angular momentum. The object which is forming under this condition is a vortex, which we will describe in more detail in the next sections.
The condition for irrotationality for a system with a vortex along the z-axis can be generalized to

$$\boldsymbol{\nabla} \times \mathbf{v} = \hat{z}\frac{lh}{m}\delta^2(\rho)\,, \tag{4.9}$$

with the two-dimensional delta function in the xy-plane, and $\hat{z}$ being the unit vector along the z direction.

4.3 Azimuthal velocity field of a vortex

The most simple example of a vortex is given by a purely azimuthal velocity in a trap which is rotationally invariant along the z axis. Due to its single valuedness, the wavefunction must contain a phase factor $e^{il\varphi}$, where φ is the azimuthal angle. We can use expression (4.6) to calculate the azimuthal velocity v_φ in cylindrical coordinates,

$$v_\varphi = \frac{\hbar}{m}\nabla\phi = \frac{\hbar}{m}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi \tag{4.10}$$

$$= \frac{\hbar}{m}\frac{l}{\rho}\,. \tag{4.11}$$

Obviously, the circulation is lh/m if the contour C encloses the axis, but zero if it does not.

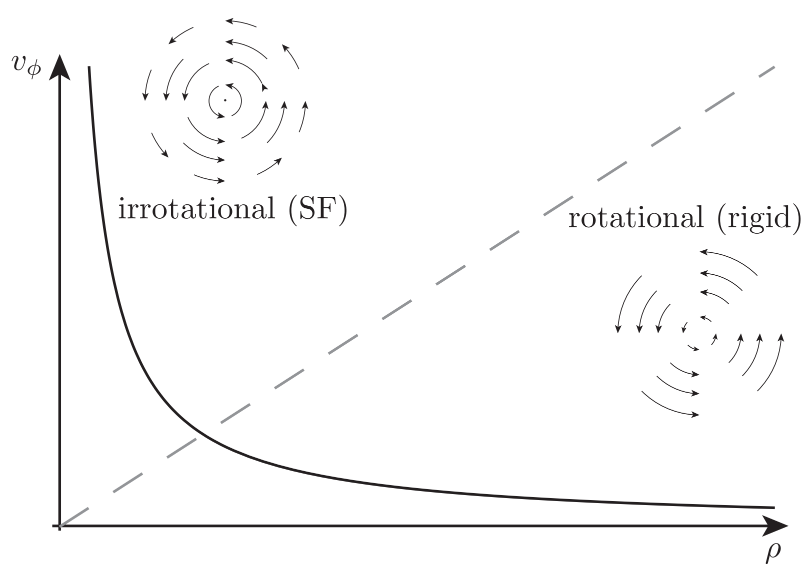


Figure 4.1: **Azimuthal velocity of a vortex.** The velocity field for a normal rotating fluid behaves like a rigid body, where the azimuthal velocity increases linearly with increasing distance from the symmetry axis. A superfluid shows a non-intuitive velocity pattern, where the azimuthal velocity diverges for decreasing distance from the rotation axis. In order not to let the kinetic energy of the system diverge, the density of the superfluid must go to zero for $\rho \to 0$.

The azimuthal velocity for $l \neq 0$ is proportional to $1/\rho$, as shown in figure 4.1 in contrast to

a rigid body. This implies that the kinetic energy of the system would diverge in presence of a vortex. The way out is to force the density $|f|^2$ of the condensate to go to zero for $\rho \to 0$. This now already gives us an intuitive picture of a vortex, which has a core with zero density. Accordingly, in a 3D system a vortex line will form.
The total angular momentum L carried by the whole condensate with N particles in presence of a vortex at the symmetry axis with winding number l is $Nl\hbar$, i.e. each particle carries $l\hbar$ angular momentum.

4.4 The structure of a single vortex

We now want to get a better understanding of the density structure of a vortex. Until now we only know that the density has to vanish when approaching the vortex line. We use cylindrical coordinates and put the Ansatz

$$\psi(\mathbf{r}) = f(\rho, z)e^{il\varphi} \tag{4.12}$$

into the time-independent Gross-Pitaevskii equation (again in cylindrical coordinates), which results in

$$-\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial f}{\partial\rho}) + \frac{\partial^2 f}{\partial z^2}\right) + \frac{\hbar^2 l^2}{2m\rho^2}f + V(\rho, z)f + U_0 f^3 = \mu f\,. \tag{4.13}$$

Here we used the Laplacian in cylindrical coordinates and already carried out the second derivative with respect to φ. The resulting term $\frac{\hbar^2 l^2}{2m\rho^2}$ is giving the kinetic energy due to the azimuthal velocity field of a vortex. This term is sometimes also called the centrifugal barrier of the problem. For a superfluid with $l = 0$, this term vanishes and the above equation is the standard GPE.
As a first step we try to estimate the typical size of a vortex. We consider a uniform system (i.e. $V = 0$, derivative with respect to z vanishes) with a rotating condensate with $l = 1$, which will form a vortex line. We know that far away from the vortex line, the system behaves as a non-rotating superfluid, such that for large ρ the centrifugal barrier term $\propto 1/\rho^2$ and the derivative with respect to ρ becomes unimportant. Neglecting also these terms in equation (4.13) we see that the wave function f approaches the wave function $f_0 = \sqrt{\mu/U_0}$ of a non-rotating condensate.
In contrast, for small distance ρ from the vortex line, the terms proportional to $1/\rho$ will dominate. We make the ansatz $f \propto \rho^l = \rho$ as known from free particles with angular momentum. This ansatz also ensures that the superfluid density goes to zero when approaching $\rho = 0$. Then the terms dominating the small ρ behavior will cancel. To find the cross over between this and the large ρ behavior, and thus the typical size of a vortex, we make the simple argument that kinetic energy should equal interaction energy. For a typical size ξ of the vortex we then get

$$-\frac{\hbar^2}{2m\xi^2} = U_0 n\,, \tag{4.14}$$

which leads to to the typical size

$$\xi \approx \sqrt{\frac{\hbar^2}{2mU_0 n}}\,. \tag{4.15}$$

This length scale is also called the healing length of the condensate. It describes the typical distance over which the condensate wave function tends to its bulk value if there is a localized perturbation present, and is called healing length.
Instead of estimating the typical size and doing approximations, we can also make use of the Gross-Pitaevskii equation and directly try to solve for the density profile of a vortex. It is

convenient to rescale variables in terms of the healing length ξ, and the unperturbed condensate wave function $\tilde{\psi}_0$, such that we have

$$x = \frac{\rho}{\xi} \tag{4.16}$$

$$\chi = \frac{f}{f_0}. \tag{4.17}$$

The GPE with these rescaled variables then reads

$$-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\chi}{\mathrm{d}x}\right) + \frac{\chi}{x^2} + \chi^3 - \chi = 0 \tag{4.18}$$

which can be solved numerically. The solutions for $l = 0, 1, 2$ are plotted in figure 4.2, and show the same behavior which we derived before using our approximations and assumptions.

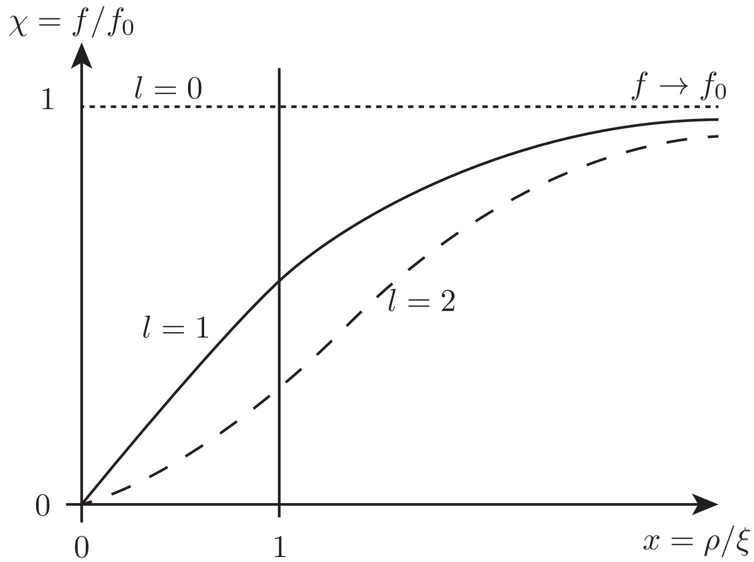


Figure 4.2: **Density profile of a vortex.** Shown are the density profiles for the system without a vortex ($l = 0$), and for a singly and doubly charged vortex ($l = 1, 2$). The short-range behavior is given by $f \propto \rho^l$, while the asymptotic behavior of the wave function always approaches the wave function of the unperturbed system. The cross over between the short-range and the long-range behavior takes place around the healing length $\rho = \xi$.

4.5 The energy of a vortex

We will now consider the energy of a vortex for a system contained in a cylinder with radius $D \gg \xi$. We have to take this assumption of a finite system, as the energy for an infinite system would diverge logarithmically. The resulting energy can be approximated by subtracting the energy of a system without a vortex from the energy of a system with a vortex. The resulting estimate for the energy of a vortex per unit length,

$$\epsilon_v \approx l^2 \pi n \frac{\hbar^2}{m} \ln\left(\frac{D}{\xi}\right), \tag{4.19}$$

has been confirmed by numerically solving the corresponding Gross-Pitaevskii equation, which yields $\epsilon_v \approx l^2 \pi n \frac{\hbar^2}{m} \ln\left(1.46\frac{D}{\xi}\right)$. This energy scales with the square of the winding number l. This result indicates that the system will actually prefer a state with several well-separated vortices with $l = 1$ with respect to one highly charged vortex.

Instead of calculating the energy in the laboratory frame, we can also move into the rotating frame, and calculate the energy using a Galilean transformation

$$E'_v = E_v - \Omega L_z, \tag{4.20}$$

where L_z refers to the angular momentum of the condensate in the laboratory frame. We see now that in the rotating frame the solution with $\Omega L_z > 0$ becomes energetically favorable with respect to the ground state with $L_z = 0$ if the angular velocity Ω is high enough, exceeding a critical angular velocity Ω_c which depends on the details of the system.