Chapter 2

Scaling theory of localization

Learning goals

- We appreciate the role of the dimensions for the localization of electrons.
- We can reproduce the gang-of-four scaling plot.

- M. A. Paalanen G. A. Thomas, Helv. Phys. Acta **56**, 27 (1983)

2.1 Conductance versus conductivity

We want to study the influence of disorder on the electrical resistance R relating the applied voltage U to the electrical current I

$$U = RI. \tag{2.1}$$

R connects two macroscopic observables and therefore characterizes a macroscopic sample. The conductance g is defined as the inverse of the resistance

$$g = \frac{1}{R}. \tag{2.2}$$

These quantities have to be contrasted with the microscopic quantities such as the conductivity σ

$$j = \sigma E, \tag{2.3}$$

where E is the electric field and j the microscopic current density. In this chapter we want to understand if there is a simple bridge between the microscopic quantity σ (which we might be able to calculate from first principles in simple model situations) and the macroscopic conductance g. We try to do so by starting from a relatively small system where we are in principle up to the task of calculating g exactly. We then want to successively increase the system size and see what we can deduce.

2.2 One parameter scaling

The key step in the program of successively increasing the system size dates back to the very influential paper by what we now call the the "gang of four": Abrahams, Anderson, Licciardello, and Ramakrishnan [1]. Their key insight was that the conductance $g(2L)$of a block of size $2L$ only depends on one parameter, namely the conductance $g(L)$ of the block of size L out of which the larger was formed, cf. Fig. 2.1. In other words

$$g(2L) = f[g(L)] \qquad \text{and not} \qquad g(2L) = h[g(L), L, \dots]. \tag{2.4}$$

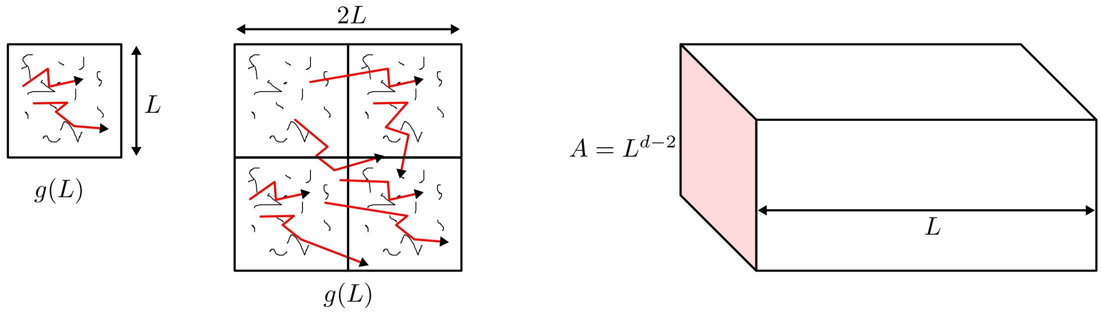


Figure 2.1: Setup for the renormalization of the conductance.

This statement is not easy to motivate in a systematic way. Instead of attempting to legitimate (2.4), we want to analyze its consequences in the following. To make further progress we write (2.4) in a form that contains no scales

$$\frac{L}{g}\frac{dg(L)}{dL} = \frac{d\log(g)}{d\log(L)} = \beta(g). \tag{2.5}$$

Let us have a look at simple limiting cases.

For a good conductor $g \gg 1$ we know that the "one parameter scaling" holds in the form of Ohm's law

$$R = \rho\frac{L}{A} = \rho\frac{L}{L^{d-1}} \qquad \Rightarrow \qquad g = \sigma_0 L^{2-d}. \tag{2.6}$$

From this we immediately obtain

$$\frac{d\log(g)}{d\log(L)} = d-2 \qquad \Rightarrow \qquad \lim_{g\to\infty}\beta(g) = d-2. \tag{2.7}$$

In the other limit of very strong disorder, all wave-function will be exponentially localized. Therefore, we expect the conductance to behave as

$$g(L) \propto e^{-L/\xi} \qquad \Rightarrow \qquad \frac{d\log(g)}{d\log(L)} = -\frac{L}{\xi} = \log(g). \tag{2.8}$$

Hence in the limit of vanishingly small conductance, the β function reads

$$\lim_{g\to 0}\beta(g) = \log(g). \tag{2.9}$$

We summarize these results in Fig. 2.2. Due to the dependence of the β-function on the dimension d, disorder seems to have very different effects depending on the spatial dimension. Let us discuss the consequences of Fig. 2.2 for one, two, and three dimensions separately.

2.2.1 One dimension

In one dimension, $\beta(g) < 0$ is always negative. In other words, by increasing the system size, the conductance g always flows to zero, irrespective of the conductance of a short section of the wire.

Let us define a localization length ξ, where $g(L = \xi) = 1$. We find

$$\frac{d\log(g)}{d\log(L)} = -1 \qquad \Rightarrow \qquad g(L) = \frac{g_0}{L} \qquad \Rightarrow \qquad \xi \sim g_0, \tag{2.10}$$

where g_0 is the conductance calculated for a small segment.

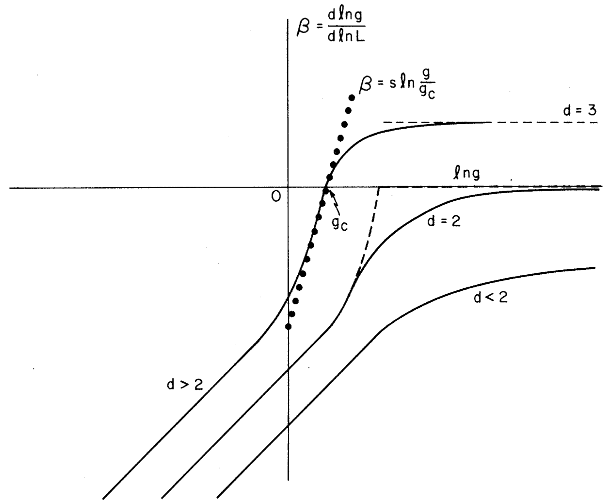


Figure 2.2: Plot of $\beta(g)$ as a function of $\log(g)$ for various dimensions. Figure take from Ref. [1] (Copyright (1979) by The American Physical Society).

2.2.2 Two dimensions

In two dimensions we encounter a somewhat more intriguing situation. For large values of g the β-function is zero. In other words, to first order in $1/g$, the conductance does not change under a change in L. Such a situation is called marginal. As we have identified the limit $g \gg 1$ as the classical regime where Ohm's law holds, this means quantum corrections will play a crucial role in how $\beta(g)$ behaves away from $g \gg 1$. These quantum corrections are called "weak (anti-) localization". Their detailed calculation is beyond the scope of this course. However, we can estimate them using a simple trick. Let us just calculate the probability for a particle to return to the point where we inject it into the system

$$P = |\langle \psi^\dagger(x)\psi(x)\rangle|^2. \tag{2.11}$$

When we calculate $\langle \psi^\dagger(x)\psi(x)\rangle$, we have to sum over all path the particle can take from x, back to the same point x. In quantum mechanics, each path is associated with an amplitude and a phase. Due to the disorder (which we try to study, after all), all paths sum up incoherently. If we have a time-reversal invariant system, however, there are paths who's amplitude and phase are correlated as shown in Fig. 2.3. Owing to the time-reversal symmetry the blue and the red path have a well defined phase relation. If we now calculate P the sum contains the following contributions shown in Fig. 2.4.

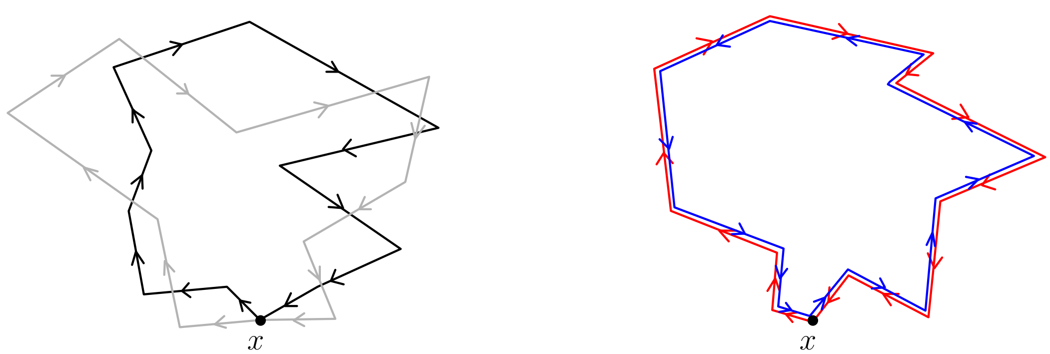
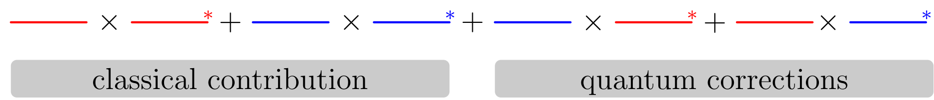


Figure 2.3: Return probability.



The endpoints of any segment are related to some state $|\phi\rangle$. In order to invert the arrow of time we use the time reversal operator $\mathcal{T}$ on these states:

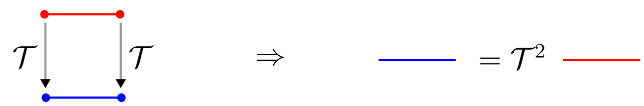


Figure 2.4: Interference in the return probability.

From this we conclude that we can have two distinctly different situations

1. $\mathcal{T}^2 = -\mathbb{1}$ $\Rightarrow$ the return probability P is reduced, hence quantum mechanical effects lead to more extended states and we deal with weak *anti-localzation*.

2. $\mathcal{T}^2 = \ \ \mathbb{1}$ $\Rightarrow$ the return probability P is enhanced, i.e., the states are more localization: weak *localization*.

However, a word of caution is in order here! When applying this argument for spin-1/2 fermions we generically have $\mathcal{T}^2 = -\mathbb{1}$. But if the Hamiltonian does not mix the spin degrees of freedom, we can go to the individual spin sectors and describe the physics as two spin-0 problems. In this case however, $\mathcal{T}^2 = \mathbb{1}$. The situation changes if we deal with *spin-orbit* coupling. In this case we have to stick with the spin-1/2 description and therefore we generically expect *anti-localization* in this case.

Let us now analyze the case of no spin-orbit interactions, i.e., weak localization. We solve the equation

$$\frac{d\log(g)}{d\log(L)} = -\frac{C}{g} \qquad \text{or} \qquad \frac{dg}{d\log(L)} = -C < 0. \tag{2.12}$$

We find

$$g = g_0 - C\log(L/l), \tag{2.13}$$

where l is the small length at which we managed to solve the problem exactly and found $g(l) = g_0$. We can now again determine the localization length by equating $g(\xi) = 1$ to find

$$\xi \sim l e^{g_0/C}. \tag{2.14}$$

Indeed, all states are localized. However, $g \gg g_0$ and the localization length is astronomical.

2.2.3 Three dimensions

Three dimensions (or two with spin-orbit) are the most interesting cases. Depending on the initial value g_0, $\beta(g)$ is either positive or negative, i.e., a macroscopic sample can either be conducting or insulating. In other words, there is a metal-insulator transition as a function of g_0. For the time being three dimensions are not in the scope of the course and we will come back to it (and spin-orbit in two dimensions) later.

References

1. Abrahams, E., Anderson, P. W., Licciardello, D. C. & Ramakrishnan, T. V. "Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions". *Phys. Rev. Lett.* **42,** 673 (1979).