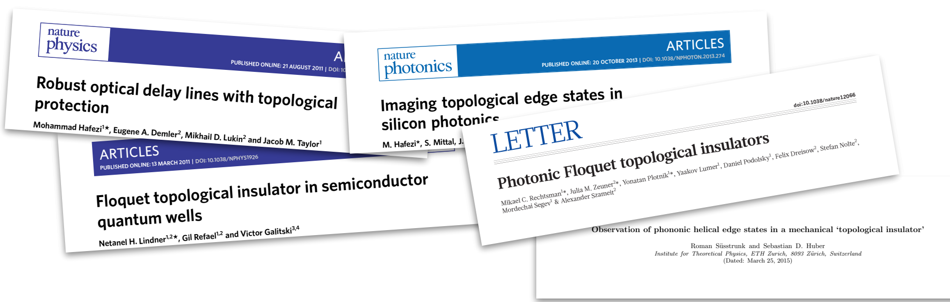
Chapter 6

Significant others



Learning goals

- We know the basic phenomenology of the quantum Hall effect (QHE)
- We know the structure of the lowest Landau level (LLL)
- We understand the role of disorder for the QHE.

- K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980)

6.1 Introduction

Up to now we were dealing with topological effects in electronic systems. We started with the quantum Hall effect and have seen that many more variants of electronic systems can have topological band structures. In the present chapter we want to see how the ideas of topological band theory found applications in other, seemingly unrelated, fields such as photonic or mechanical systems.

This chapter is also somewhat unusual as we follow a different approach in how the topic is being taught. We will see that among the systems in which topological bands can show up are (i) coupled ring resonators, (ii) evanescently coupled optical wave guides, (iii) periodically driven systems (electronic or photonic), and (iv) idealized mechanical systems. In the lecture we provide a few necessary ingredients to understand the core aspects of these physical systems. These prerequisites should then allow you to understand the original research papers we list at the end of this chapter covering the topics (i) – (iv).

6.2 Photonic systems

6.2.1 Coupled ring resonators

This section follows closely Ref. [1]. One can create "tight-binding" models for optical systems by weakly coupling high-Q resonators. Similar to the tight-binding approach for electronic band structures one assumes the electromagnetic field to be well described by the mode of a single high-Q cavity $\mathbf{E}_\Omega(\mathbf{r},t)$ with frequency Ω. When several cavities are weakly coupled, for example by evanescent waves, the electric field will largely be described by the local cavity fields $\mathbf{E}_\Omega(\mathbf{r},t)$ plus a small correction due to their coupling. In other words, the local cavities play the roles of the atomic orbitals and the evanescent coupling corresponds to the tunneling matrix element between these orbitals.

We assume a one dimensional chain of coupled cavities separated by a length a in $\hat{\mathbf{e}}_x$ direction for simplicity. Let us write for the electric field of the full problem

$$\mathbf{E}_k(\mathbf{r},t) = e^{i\omega_k t}\sum_{n\in\mathbb{Z}} e^{-ikna}\,\mathbf{E}_\Omega(\mathbf{r}-na\hat{\mathbf{e}}_x). \tag{6.1}$$

We readily recognize $\mathbf{E}_\Omega(\mathbf{r})$ as the Wannier function and $\mathbf{E}_k(\mathbf{r},t)$ as the corresponding Bloch "wave"-function. Recall that in the absence of source terms the Maxwell equations read

$$\nabla\wedge\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{6.2}$$

$$\nabla\wedge\mathbf{B} = \mu_0\epsilon(\mathbf{r})\frac{\partial\mathbf{E}}{\partial t}, \tag{6.3}$$

where μ_0 is the vacuum permeability and $\epsilon(\mathbf{r})$ the dielectric constant of the coupled resonators. The above equations can be combined to

$$\nabla\wedge[\nabla\wedge\mathbf{E}_k(\mathbf{r})] = \mu_0\epsilon(\mathbf{r})\omega_k^2\mathbf{E}_k(\mathbf{r}). \tag{6.4}$$

We assume $\mathbf{E}_\Omega(\mathbf{r})$ to be real and normalized to

$$\int d\mathbf{r}\,\epsilon(\mathbf{r})\mathbf{E}_\Omega(\mathbf{r})\cdot\mathbf{E}_\Omega(\mathbf{r}) = 1. \tag{6.5}$$

$\mathbf{E}_\Omega(\mathbf{r},t)$ satisfies the same equation with $\epsilon(\mathbf{r})$ replaced by $\epsilon_0(\mathbf{r})$, the dielectric constant of a single resonator. If we insert (6.1) into (6.4) we obtain

$$\omega_k^2 = \Omega^2\frac{1+\sum_{n\neq 0}\exp(-ikna)\beta_n}{1+\Delta\alpha\sum_{n\neq 0}\exp(-ikna)\alpha_n}, \tag{6.6}$$

where

$$\alpha_n = \int d\mathbf{r}\,\epsilon(\mathbf{r})\mathbf{E}_\Omega(\mathbf{r})\cdot\mathbf{E}_\Omega(\mathbf{r}-na\hat{\mathbf{e}}_x), \tag{6.7}$$

$$\beta_n = \int d\mathbf{r}\,\epsilon_0(\mathbf{r}-na\hat{\mathbf{e}}_x)\mathbf{E}_\Omega(\mathbf{r})\cdot\mathbf{E}_\Omega(\mathbf{r}-na\hat{\mathbf{e}}_x), \tag{6.8}$$

$$\Delta\alpha = \int d\mathbf{r}\,[\epsilon_0(\mathbf{r})-\epsilon(\mathbf{r})]\mathbf{E}_\Omega(\mathbf{r})\cdot\mathbf{E}_\Omega(\mathbf{r}). \tag{6.9}$$

For the case of weakly coupled resonators we can assume that $\alpha_n = \beta_n = 0$ for $n \neq 1,-1$. Moreover, symmetry requires $\alpha_1 = \alpha_{-1}$ and $\beta_1 = \beta_{-1}$. Putting these observations together we find

$$\omega_k = \Omega\left[1-\frac{\Delta}{2}+\kappa_1\cos(ka)\right], \tag{6.10}$$

where

$$\kappa_1 = \beta_1 - \alpha_1 = \int d\mathbf{r}\,[\epsilon_0(\mathbf{r} - a\hat{\mathbf{e}}_x) - \epsilon(\mathbf{r} - a\hat{\mathbf{e}}_x)]\mathbf{E}_\Omega(\mathbf{r}) \cdot \mathbf{E}_\Omega(\mathbf{r} - na\hat{\mathbf{e}}_x). \tag{6.11}$$

From these considerations one can clearly see that the system of coupled resonators is described exactly like a tight-binding hopping model for electrons.
In the papers by Hafezi et al. [2, 3] there is a small twist on the setup presented here. The "hopping" is not merely through evanescent coupling, but via the help of another wave-guide, or cavity, of variable length. Read the papers [2] and [3] and try to understand how their setup connects to what we learned in the last chapter.

6.2.2 Paraxial wave equation

Let us assume the wave equation describing the evolution of the electric field $\mathbf{E}(\mathbf{r}, t) = e^{i\omega t}\mathbf{E}(\mathbf{r})$

$$\nabla^2\mathbf{E}(\mathbf{r}) = -\omega^2\mu_0\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}). \tag{6.12}$$

We are now making the assumption that the electro-magnetic wave propagate mainly in z-direction and can be written as

$$\mathbf{E}(\mathbf{r}) = E_0(\mathbf{r})e^{ik_0 z}\hat{\mathbf{e}}_x, \tag{6.13}$$

where $E_0(\mathbf{r})$ is a slowly varying envelope that fulfills

$$|\partial_z^2 E_0(\mathbf{r})| \ll |k_0\partial_z E_0(\mathbf{r})|. \tag{6.14}$$

Hence, we can write (6.12) as

$$i\partial_z E_0(x, y, z) = -\frac{1}{2k_0}\left[\partial_x^2 + \partial_y^2 + \omega^2\mu_0\epsilon(x, y, z)\right] E_0(x, y, z). \tag{6.15}$$

Imagine now a periodic collection of parallel wave guides running along the z-direction. In other words, $\epsilon(x, y, z) = \epsilon(x, y)$ with $\epsilon(x+na, y+mb) = \epsilon(x, y)$ for some $a, b \in \mathbb{R}$ and $n, m \in \mathbb{Z}$. Hence, we can think of the z-direction as the time axis. The transverse dynamics is then equivalent to the one governed by the single particle Schrödinger equation in the periodic potential $\epsilon(x, y)$.
These considerations are a first step in understanding the "Photonic Floquet topological insulator" presented in Ref. [4]. Before we can embark on this paper, however, we need a few more prerequisites.

6.2.3 Floquet theory

A system which is topologically trivial might be rendered non-trivial by the application of a periodic perturbation. In order for us to apply our toolbox for the classification of topological insulators we need to have a time-independent problem at hand. Owing to the periodic nature of the perturbation, we can use the time-domain analog to Bloch theory: Floquet theory [5]. The basic observation is that solutions of the Schrödinger equation of a Hamiltonian of the form $H(t) = H(t + T)$ can be written as

$$\psi(t) = e^{i\epsilon_\alpha t}\Phi_\alpha(t) \qquad \text{with} \qquad \Phi_\alpha(t) = \Phi_\alpha(t + T). \tag{6.16}$$

Clearly we observe that ϵ_α, called the quasi-energy, is the counterpart to the quasi-momentum in Bloch theory. Moreover, if we are only interested in the "stroboscopic" evolution to times $t = nT$, with $n \in \mathbb{Z}$, the wave-function changes by

$$\psi(T) = e^{i\epsilon_\alpha T}\Phi_\alpha(0). \tag{6.17}$$

It is easy to show that $\Phi_\alpha(t)$ are the solutions to

$$[H(t) - i\hbar\partial_t]\Phi_\alpha(t) = \mathcal{H}_F(t)\Phi_\alpha(t) = \epsilon_\alpha\Phi_\alpha(t). \tag{6.18}$$

$\mathcal{H}_F$ is called the Floquet operator. Furthermore, we can write for the evolution operator

$$K(T) = \mathcal{T} \exp\left[-\frac{i}{\hbar}\int_0^T dt\, H(t)\right] = \exp\left[\frac{i}{\hbar} H_F\right], \tag{6.19}$$

where $\mathcal{T}$ is time time-ordering operator. The Floquet Hamiltonian H_F is time-independent, has the eigenvalues ϵ_α, and governs the topological properties at long times.[1] The standard task in Floquet theory is to find the effective time-independent H_F corresponding to the time-dependent Floquet operator $\mathcal{H}_F$.
A complete coverage of Floquet theory is well beyond the scope of this lecture. However, there are two limiting cases where we can understand what has to be done.

The resonant case

First, in the limit where the drive frequency ω is (near) resonant with a level spacing of the Hamiltonian we can apply the rotating wave approximation (RWA). Let us consider a simple example

$$H(t) = \frac{\hbar\omega_0}{2}\sigma_z + \lambda\cos(\omega t)\sigma_x. \tag{6.20}$$

We can go to a rotating frame by

$$U = P_+ + P_- e^{i\omega t} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{i\omega t}. \tag{6.21}$$

We now transform

$$U^\dagger \mathcal{H}_F U = \underbrace{\begin{pmatrix} \frac{\hbar\omega_0}{2} & \frac{\lambda}{2} + \frac{\lambda}{2}e^{i2\omega t} \\ \frac{\lambda}{2} + \frac{\lambda}{2}e^{-i2\omega t} & \hbar\omega - \frac{\hbar\omega_0}{2} \end{pmatrix}}_{H_F} - i\hbar\partial_t. \tag{6.22}$$

If $\omega \approx \omega_0$ and $\lambda \ll \hbar\omega_0$, the two states are approximately degenerate and we can neglect the term $\propto \exp(2i\omega t)$ as it quickly averages to zero. Hence, we found a time-independent Hamiltonian H_F. This approximation is relevant for a resonant drive as discussed in Ref. [6].

The off-resonant case

In the second limit, where ω is much larger than any frequency in the Hamiltonian we can find H_F in a series in $1/\omega$. To this end, we expand the time-ordered product in (6.19)

$$H_F^0 = \frac{1}{T}\int_0^T dt H(t), \qquad H_F^1 = -\frac{i}{2T}\int_0^T dt \int_0^t dt'\,[H(t), H(t')]. \tag{6.23}$$

We now write $H(t)$ as a Fourier-series ($\omega = 2\pi/T$)

$$H(t) = \sum_{n\in Z} H_n e^{i\omega n t}, \tag{6.24}$$

which leads to

$$H_F^0 = H_0, \tag{6.25}$$

$$H_F^1 = \frac{1}{\omega}\sum_{n=1}^{\infty}\frac{1}{n}\left([H_n, H_{-n}] - [H_n, H_0] + [H_{-n}, H_0]\right). \tag{6.26}$$

Applying this procedure to graphene in a circularly polarized electric field, one can understand both Refs. [4] and [7].

[1] Here, we gallantly overlooked how this procedure depends on the initial time t (which we set to zero) and the phase of the periodic driving.

6.3 Publications

Study the following papers in groups of two to three students and explain them in two weeks in a short presentation:

- Quantum spin Hall effect in silicon photonics [2, 3].
- Floquet topological insulators in semi-conductors [6].
- Photonic topological insulator in coupled wave-guides [4].
- Quantum spin Hall effect for phonons [8].

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