Chapter 7

The fractional quantum Hall effect I

Learning goals

- We are acquainted with the basic phenomenology of the fractional quantum Hall effect.
- We know the Laughlin wave function.
- We can explain the mutual statistic of Laughlin quasi-particles

- D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982)

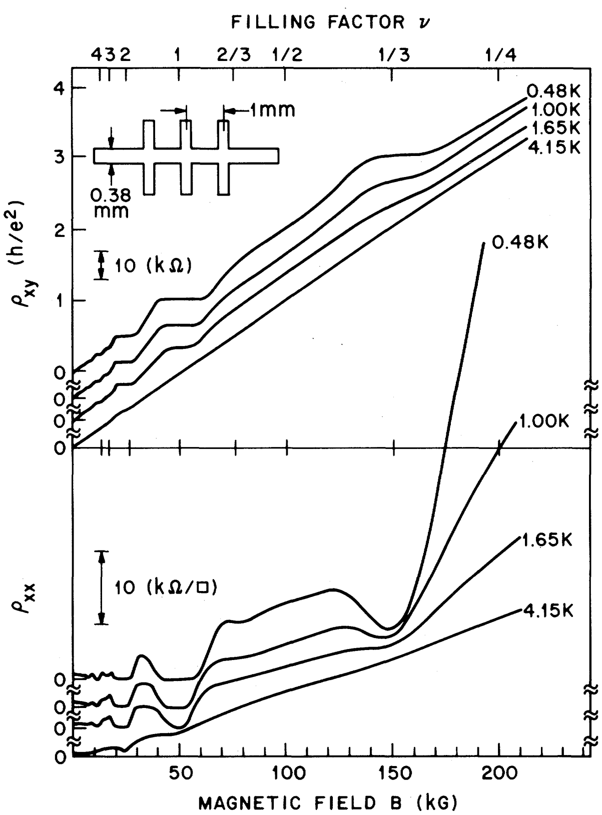

Figure 7.1: Measurements of the longitudinal and transverse resistance in a semiconductor heterostructure. At low temperatures a Hall plateau develops at a filling fraction $\nu = 1/3$ together with a dip in the transverse conductance. Figure take from Ref. [1] (Copyright (1982) by The American Physical Society).

We have seen that the Hall conductance in a large magnetic field is quantized to multiples of the quantum of conductance e^2/h. We could explain this quantization via a mapping of the linear response expression for the Hall conductance to the calculation of the Chern number of ground state wave function. The seminal experiment of Tsui et al. showed, however, that in a very clean sample, the Hall conductance develops a fractional plateau at one third of a quantum of conductance, see Fig. 7.1. In this chapter we try to understand how this can come about and how it is compatible with our derivation of the integer-quantized Hall conductance. So far we have only dealt with free fermion systems where the ground state was a Slater determinant of single particle states. Let us start from such a ground state and see how we might understand the fractional quantum Hall effect via a wave function inspired by such a Slater determinant.

7.1 Many particle wave functions

We have seen in the exercise class that in the symmetric gauge, where $\mathbf{A} = -\frac{1}{2}\mathbf{r} \wedge \mathbf{B}$, the lowest Landau level wave function can be written as

$$\psi_m(z) \propto z^m e^{-\frac{1}{4}|z|^2}, \qquad z = \frac{1}{l}(x+iy), \qquad l = \sqrt{\frac{\hbar}{eB}}. \tag{7.1}$$

We have also seen that the m'th wave function is peaked on a ring that encircles m flux quanta. A direct consequence of (7.1) is that any function

$$\psi(z) = f(z) e^{-\frac{1}{4}|z|^2} \tag{7.2}$$

with an analytic $f(z)$ is in the lowest Landau level. Let us make use of that to address the many-body problem at fractional filling. At fractional fillings, these is no single-particle gap as the next electron can also be accommodated in the same, degenerate, Landau level. Hence, we need interactions to open up a gap. Let us assume a rotational invariant interaction, e.g., $V(r) = e^2/\epsilon r$. Moreover, we start with the two-particle problem. Requiring relative angular momentum m and total angular momentum M, the only *analytic* wave function is

$$\psi_{m,M}(z_1, z_2) = (z_1 - z_2)^m (z_1 + z_2)^M e^{-\frac{1}{4}\left(|z_1|^2 + |z_2|^2\right)}. \tag{7.3}$$

Given the azimuthal part (angular momentum), no radial problem had to be solved! The requirement to be in the lowest Landau level fixes the radial part. ⇒ All we need to know about $V(r)$ are the Haldane pseudo-potentials[1]

$$v_m = \langle Mm|V|Mm\rangle. \tag{7.4}$$

7.1.1 The quantum Hall droplet

Let us now construct the many-body state for the two-particle state centered around $z = 0$. For $\nu = 1$ we construct the Slater determinant with the orbits $m = 0, 1$

$$\psi(z_1, z_2) = f(z_1, z_2) e^{-\frac{1}{4}\sum_{j=1}^{2}|z_j|^2} \qquad \text{with} \qquad f(z_1, z_2) = \begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix} = -(z_1 - z_2). \tag{7.5}$$

The generalization to N particles with $m = 0, \ldots, N-1$ will fill a circle of radius $\sqrt{2N}$ and f is given by the Vandermonde determinant

$$f = -\prod_{i<j}(z_i - z_j). \tag{7.6}$$

[1] If we neglect Landau level mixing!

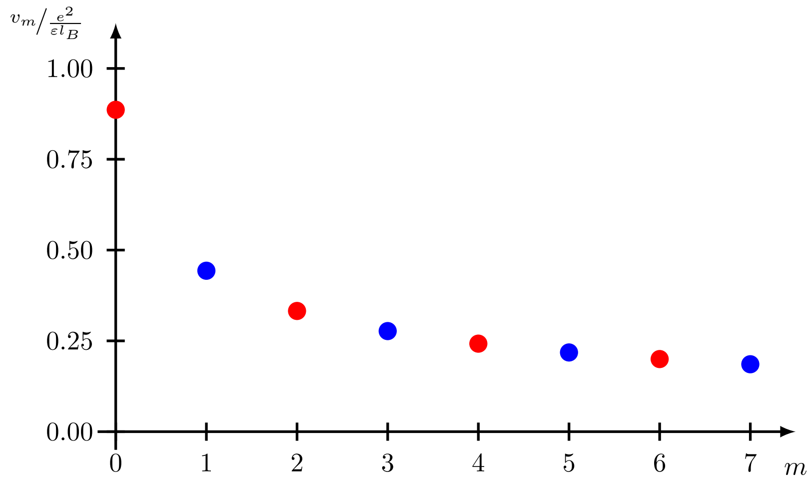

Figure 7.2: Haldane pseudo potentials for the Coulomb interaction in the lowest Landau level as a function of relative angular momentum m. The even relative angular momenta (red) are irrelevant for a fermionic system. In the following we approximate the full Coulomb potential with the first pseudo potential by setting $v_{m>1} \equiv 0$

Therefore, the many-body wave function of a filled lowest Landau level is given by

$$\psi(\{z_i\}) = \prod_{i<j}(z_i - z_j)e^{-\frac{1}{4}\sum_{j=1}^N |z_j|^2}. \tag{7.7}$$

Building on this form of the ground state wave function R. Laughlin made the visionary step [2] of proposing the following wave function for the one third filled Landau level[2]

$$\psi_{\rm L}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^3 e^{-\frac{1}{4}\sum_{j=1}^N |z_j|^2}. \tag{7.8}$$

Before we embark on a detailed analysis of this wave function, let us make a few simple comments: (i) No pair of particles has a relative angular momentum $m < 3$! $\Rightarrow$ if we only keep the smallest non-trivial Haldane pseudo potential v_1, $\psi_{\rm L}$ is an exact ground state wave function in the lowest Landau level. (ii) if $g(\{z_i\})$ is a symmetric (under exchange $i \leftrightarrow j$) polynomial, then $\psi = g\psi_{\rm L}$ is also in the lowest Landau level. In particular

$$\psi_{\{w_s\}}(\{z_i\}) = \prod_{s=1}^{n}\prod_{j=1}^{N}(z_j - w_s)\psi_{\rm L}(\{z_i\}) \tag{7.9}$$

is a wave function of N particles depending on the n (two dimensional) parameters $w_n = x_n + iy_n$ and is in the lowest Landau level. We will study this generalization if the Laughlin wave-function in the following. Keep in mind that the ground-state shall be described by $\psi_{\rm L}(\{z_i\})$ and we will argue that $\psi_{\{w_s\}}(\{z_i\})$ corresponds to an excited state with quasi-holes at the positions w_s.

7.2 The plasma analogy

In order to better understand the Laughlin wave function we make use of a very helpful analogy called the "plasma analogy" [3]. We write the probability distribution in the form

$$|\psi_{\{w_s\}}(\{z_i\})|^2 = \exp\left[-6E_{\{w_s\}}(\{z_i\})\right] = e^{-\beta E}, \quad Z = \int d\mathbf{z} e^{-\beta E}, \tag{7.10}$$

[2]It is maybe interesting to state here the *full* abstract of this paper: *This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.* Keep its length in mind when you write your Nobel paper...

with

$$E_{\{w_s\}}(\{z_i\}) = -\frac{1}{3}\sum_{sj}\log|z_j - w_s| - \sum_{i<j}\log|z_i - z_j| + \sum_j \frac{|z_j|^2}{12}. \tag{7.11}$$

We will argue in the following that $|\psi_{\{w_s\}}(\{z_i\})|^2$ is given by the Boltzmann weight if a *fake* classical plasma at inverse temperature $\beta = 6$. Note that this is just a way of interpreting a quantum mechanical wave function. There is no plasma involved. Moreover, when we speak of "charges" in the following, we mean the fake charges of our plasma analogy. When we are interested in real, electronic charges, we will calculate (electron) densities with the help of the plasma analogy. From these real electron densities we will infer the actual real charge.
Let us remind ourselves of two-dimensional electrodynamics. From Gauss' law we find

$$\int d\mathbf{s}\,\mathbf{E} = 2\pi Q \quad \Rightarrow \quad \mathbf{E}(\mathbf{r}) = \frac{Q\hat{\mathbf{r}}}{r} \quad \Rightarrow \quad \phi(\mathbf{r}) = -Q\log(r/r_0) \tag{7.12}$$

and the two dimensional Poisson equation is given by

$$\nabla\cdot\mathbf{E} = -\nabla^2\phi = 2\pi Q\delta(\mathbf{r}). \tag{7.13}$$

We can now interpret the terms in $E_{\{w_s\}}(\{z_i\})$:

1. $-\log|z_i - z_j|$: electrostatic repulsion between two unit charges (fake charges...).
2. $-\frac{1}{3}\log|z_i - w_s|$: interaction of a unit charge at z_i with a charge 1/3 at w_s.
3. $-\nabla^2|z|^2/12 = -1/3l^2 = 2\pi\rho_\text{b}$ with $\rho_\text{b} = -\frac{1}{3}\frac{1}{2\pi l^2}$. Hence, $\sum_j |z_j|^2/12$ is a background potential to keep the plasma (in the absence of w_s) charge neutral (Jellium).

With these interpretations we are in the position to analyze the properties of $\psi_{\{w_s\}}(\{z_i\})$:

1. $\log r$ – interactions make density variations extremely costly. Therefore the ground state, i.e., $\psi_\text{L}(\{z_i\})$ has uniform density:

$$\Rightarrow \rho = \frac{1}{3}\frac{1}{2\pi l^2} \Rightarrow \nu = \frac{1}{3}. \tag{7.14}$$

   This we could also have inferred from the fact that the largest monomial z_j^M appearing in $\psi_{\{w_s\}}(\{z_i\})$ has $M = 3N$. Hence, the radius of the droplet would be $\propto \sqrt{3N}$ and hence the area three times larger than for the $\nu = 1$ case.

2. Each w_s corresponds to a charge 1/3. Therefore, it will be screened by the z-Plasma with a compensating charge $-1/3$. $\Rightarrow$ each w_s corresponds to a *quasi-hole with* $e^* = -\frac{e}{3}$.

3. The plasma analogy also allows us to find to normalization of the wave function $\psi_{\{w_s\}}(\{z_i\})$:

$$\psi_{\{w_s\}}(\{z_i\}) = C\prod_{s<p}|w_s - w_p|^{1/3}\prod_{sj}(z_j - w_s)\prod_{i<j}(z_i - z_j)^3 e^{-\sum_j\frac{|z_j|^2}{4}}e^{-\sum_s\frac{|w_s|^2}{12}}. \tag{7.15}$$

   For this normalization we find a new plasma energy

$$E = -\frac{1}{9}\sum_{s<p}\log|w_s - w_p| - \frac{1}{3}\sum_{sj}\log|z_j - w_s| - \sum_{i<j}\log|z_j - z_i| + \sum_j\frac{|z_j|^2}{12} + \sum_s\frac{|w_s|^2}{36}. \tag{7.16}$$

   We see that all "forces" between w_s, z_j are mediated by two-dimensional Coulomb electrodynamics $\Rightarrow$ all forces on w_s are screened $\Rightarrow$

$$F_{w_s} = \frac{\partial\log Z}{\partial w_s} \approx 0 \qquad \text{for} \qquad |w_s - w_p| \gg 1. \tag{7.17}$$

   Hence $Z = \int d\mathbf{z}\,|\psi|^2 = \text{const}$, and we can normalize it with an appropriate C.

Before we calculate the charge of a quasi particle in another way that highlights the relation to their mutual statistics, σ_{xy}, and eventually the ground-state degeneracy on the torus, we want to convince ourselves that $\psi_{\rm L}$ is describing a ground state with a gapped excitation spectrum above it: If we want to make an *electronic* excitation we have to change the relative angular momentum by one. Therefore, we will have to pay the cost v_1 corresponding to the first Haldane pseudo potential! How did $\psi_{\rm L}$ manage to be such a good candidate wave function? One argument is due to Halperin [3]:
Fix all z_j expect for z_i. Take z_i around the whole droplet. $\psi_{\rm L}$ needs to pick up an Aharonov-Bohm phase $2\pi N/\nu = 2\pi N3$. $\psi_{\rm L}$ must also have N zeros (whenever $z_i \to z_j$) due to the Pauli principle. $\Rightarrow$ $2N$ zeros could be somewhere else, not bound to any special particle configuration (like to the coincidence of two particles as above) to pick up the proper Aharonov-Bohm phase. However, the *Laughlin wave function does not "waste" any zeros but uses them all to avoid interactions.*

7.3 Mutual statistics

We want to move the quasi-particle described by the location w_s around and see what Aharonov-Bohm and statistical phase they pick up. For this we calculate the Berry phase

$$\phi = \oint \mathcal{A}_\mu du^\mu \qquad \text{with} \qquad \mathcal{A}_\mu = i \left\langle \psi \middle| \partial_{u^\mu} \psi \right\rangle. \tag{7.18}$$

Our "slow" parameters u^μ are the x and y coordinates of the positions w_s of the quasi-holes. There is a problem with the above formula, however: At $w_s \to w_p$, the normalized $\psi_{\{w_s\}}(\{z_i\})$ is not differentiable. In order to make it differentiable we apply a gauge transformation

$$\tilde{\psi}_{\{w_s\}}(\{z_i\}) = e^{\frac{i}{3}\sum_{s<p}\arg(w_s-w_p)}\psi_{\{w_s\}}(\{z_i\}). \tag{7.19}$$

For fixed positions $\{w_s\}$ it is clear that this amounts to a simple global phase change. However, through

$$e^{\frac{i}{3}\sum_{s<p}\arg(w_s-w_p)} = \prod_{s<p}\frac{(w_s-w_p)^{1/3}}{|w_s-w_p|^{1/3}} \tag{7.20}$$

it cures the problem with differentiability for $w_s \to w_p$ and we can use (7.18) to calculate Berry phases. Note, however, that we made $\tilde{\psi}_{\{w_s\}}(\{z_i\})$ multivalued. The requirement of global integrability necessitated this step: a phenomena we saw already in the calculation of the Chern number.
The calculation of the Berry curvature is now straight forward. We use $w_s = x_s + iy_s$ and $\bar{w}_s = x_s - iy_s$ as our coordinates. Let us start with

$$\begin{aligned}
A_{\bar{w}_s} &= i\left\langle \psi \middle| \partial_{\bar{w}_s}\psi\right\rangle & (7.21)\\
&= i|C|^2 \int d\mathbf{z} \int d\bar{\mathbf{z}} \prod_{a<b}\prod_{cd}\prod_{e<f}(\bar{w}_a-\bar{w}_b)^{1/3}(\bar{w}_c-\bar{z}_d)(\bar{z}_e-\bar{z}_f)^3 e^{-\frac{\sum_g z_g\bar{z}_g}{4}} e^{-\frac{\sum_h w_h\bar{w}_h}{12}} & \\
&\quad \times \partial_{\bar{w}_s}\prod_{i<j}\prod_{kl}\prod_{m<n}(w_i-w_j)^{1/3}(w_k-z_l)(z_m-z_n)^3 e^{-\frac{\sum_o z_o\bar{z}_o}{4}} e^{-\frac{\sum_p w_p\bar{w}_p}{12}} & (7.22)\\
&= -i\frac{w_s}{12}. & (7.23)
\end{aligned}$$

For A_{w_s} we use the fact that our wave function is normalized

$$0 = \partial_{w_s}\langle\psi|\psi\rangle = \langle\partial_{w_s}\psi|\psi\rangle + \langle\psi|\partial_{w_s}\psi\rangle \qquad \Rightarrow A_{w_s} = \langle\psi|\partial_{w_s}\psi\rangle = -\langle\partial_{w_s}\psi|\psi\rangle. \tag{7.24}$$

The last term, however, is now easy to calculate as $\langle\psi|$ depends on w_s only through the exponential factor. Hence the calculation of A_{w_s} is analogous to the one of $A_{\bar{w}_s}$ and we find

$$A_{w_s} = i\frac{\bar{w}_s}{12}. \tag{7.25}$$

The Berry curvature is then given by

$$F_{w_s\bar{w}_s} = \partial_{w_s} A_{\bar{w}_s} - \partial_{\bar{w}_s} A_{w_s} = -\frac{i}{6}. \tag{7.26}$$

From this we can calculate the Berry phase for bringing the coordinate w_s around an area A

$$\varphi_A = -i \oiint_A dw_s d\bar{w}_s \, F_{w_s\bar{w}_s} = -\frac{1}{6} \oiint_A dxdy \frac{2}{l^2} = -\frac{\Phi_A}{3}, \tag{7.27}$$

where Φ_A is the magnetic flux through the area A. This confirms again the finding that each w_s in the wave-function $\psi_{\{w_s\}}(\{z_i\})$ describes a quasi-particle of charge

$$e^* = -\frac{e}{3}. \tag{7.28}$$

Note, that hand in hand with the appearance of a fractional charge e^*, we also picked up a non-trivial mutual statistics: If we move w_s once around w_p, we go back to the same wave-function up to a phase factor $\exp(2\pi i/3)$. This readily leads to a mutual statistical phase of $\exp(\pi i/3)$. Therefore our $e/3$ quasi-particles are neither bosons nor fermions but anyons with a statistical angle of $\pi/3$.

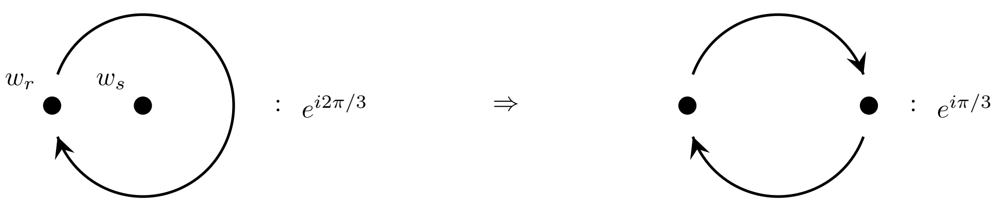

Figure 7.3: Mutual statistics.

To elucidate the connection between σ_{xy}, $e^* = -e/3$ and $\exp(i\pi/3)$ further we go through a Gedankenexperiment in analogy to Laughlin's pumping argument for the integer quantum Hall effect, cf. Fig 7.4. Let us consider a disk displaying the 1/3 fractional quantum Hall effect. We insert a flux quantum through a thin solenoid in the center. The induced current in radial direction is then given by

$$J_{\hat{\mathbf{r}}} = \sigma_{xy} E_{\hat{\boldsymbol{\varphi}}} = -\sigma_{xy} \frac{\partial \varphi}{\partial t}. \tag{7.29}$$

Therefore the charge accumulated on the center of the disk is given by

$$Q_{\text{center}} = \int dt \, J_{\hat{\mathbf{r}}} = -\frac{1}{3}\frac{e^2}{h} \int dt \, \frac{\partial \varphi}{\partial t} = -\frac{e}{3}. \tag{7.30}$$

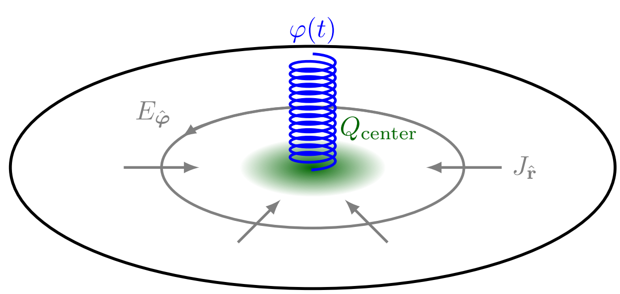

Figure 7.4: Pumping argument. Inserting a flux quantum h/e leads to an accumulation of charge $-e/3$. In the limit of an infinitely small solenoid we can gauge h/e away and we end up with a stable excitation in the form of a quasi-hole carrying one third of an electronic charge.

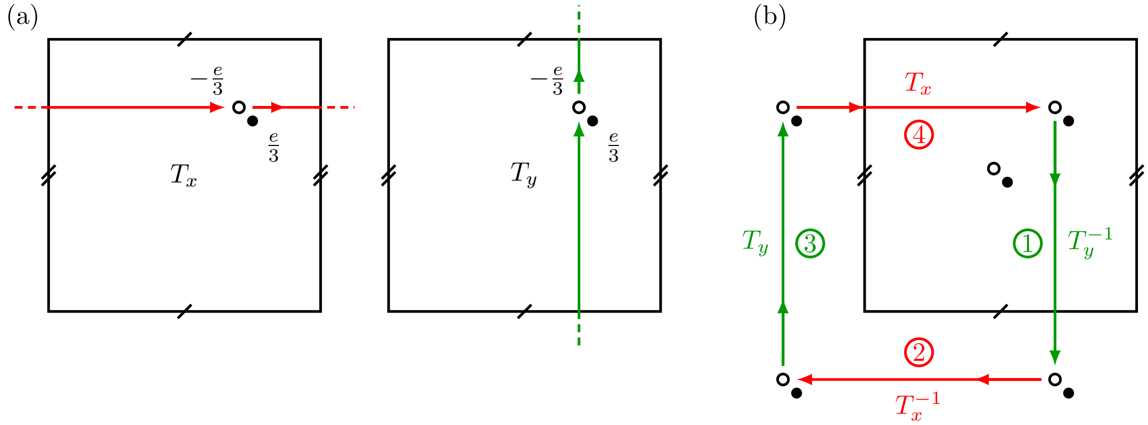

Figure 7.5: Illustration of the actions of (a) $T_{x(y)}$ and (b) $T_x T_y T_x^{-1} T_y^{-1}$ (see text).

After we inserted a full flux quantum h/e through the solenoid, we can gauge the phase away and we arrive at the same Hamiltonian. However, we do not necessarily reach the same state but we might end up in another *eigenstate* of the Hamiltonian. The accumulated charge $-e/3$ in the center must therefore be a *stable* quasi-hole after the system underwent spectral flow!
Let us bring a test quasi-hole around the solenoid: Either we think of $\exp(2\pi/3)$ as a statistical flux after we gauged away the h/e. Equivalently we can think of the additional flux of the solenoid spread over a finite area. We can then not gauge the flux away and hence we did not induce a stable quasi-hole. In contrary, the test particle accumulated a $\exp(2\pi/3)$ Aharonov-Bohm phase. This links the properties

$$\sigma_{xy} = \frac{1}{3}\frac{e^2}{h} \quad \Leftrightarrow \quad e^* = -\frac{e}{3} \quad \Leftrightarrow \quad e^{i\pi/3} - \text{anyons}. \tag{7.31}$$

7.4 Ground state degeneracy on the torus

During the discussion of the integer quantum Hall effect we found that the Hall conductivity has to be an integer multiple of e^2/h. How can we reconcile this with the fractionally quantized plateau at $\nu = 1/3$ in Fig. 7.1? The key issue was the assumption of a unique ground state on the torus with a finite gap to the first excited state. We are now proving that this is not the case of a state described by Laughlin's wave function for the $\nu = 1/3$ plateau.
Consider an operator T_x (T_y) that creates a quasi-particle – quasi-hole pair, moves the quasi-hole around the torus in x (y) direction an annihilates the two again, cf. Fig. 7.5(a). We consider now the action of $T_x T_y T_x^{-1} T_y^{-1}$. T_x shall create the pair in the middle of the chart in Fig. 7.5(b), T_y close to a corner. Moreover, the T_y movements we perform on a given chart, for the T_x movements we move the chart in the opposite direction. From this we see that one quasi-hole encircles the other! $\Rightarrow$ $T_x T_y = \exp(2\pi i/3) T_y T_x$. In addition we have the following property $T_x^3 = T_y^3 = \mathbb{1}$ as moving a full electron around the torus has to be harmless as this is what we demand for the boundary conditions.[3] The fact that $[T_x, T_y] \neq 0$ means they act on a space which is more than one-dimensional. However, they act on the ground-state manifold of the fractional quantum Hall effect on the torus. *This requires that there are several ground state sectors for the $\nu = 1/3$ state.* One can show that

$$T_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad T_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} \tag{7.32}$$

[3] Remember the gluing phase in chapter 3.

are the unique irreducible representation of the algebra defined by the above conditions. We conclude that the $\nu = 1/3$ state is threefold degenerate on the torus.
We conclude this chapter by stating that X.-G. Wen generalized the observation that ground-state degeneracy on the torus and fractional statistics are deeply linked a give rise to a new classification scheme of intrinsically topologically states (as opposed to non-interaction topological states such as the integer quantum Hall effect or more generally topological insulators) [4].

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