Week 2
Lecture Notes:
Topological Condensed Matter Physics

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Chapter 2

The integer quantum Hall effect II

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Learning goals

- We know the pumping argument of Laughlin and the concept of spectral flow.
- We know that there is always a delocalized state in each LL.
- We know that σ_{xy} is given by the Chern number.
- We understand why the Chern number is an integer.

- R. B. Laughlin, Phys. Rev. B **23**, 5632. (1981).
- D. J. Thouless, M. Kohmoto, M. Nightingale & M. den Nijs,
  Phys. Rev. Lett. **49**, 405 (1982)

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2.1 Laughlin's argument for the quantization of σ_{xy}

In the following we try to understand the pumping argument presented by R. Laughlin [1].

2.1.1 Spectral flow

The idea of *spectral flow* is central to the pumping argument of Laughlin. We try to understand this idea on the example of a particle on a ring of unit radius $r = 1$ threaded by a flux Φ[1]

$$ H = \frac{\hbar^2}{2m}\left(-\mathrm{i}\partial_\phi + \frac{\Phi}{\Phi_0}\right)^2 \quad \Rightarrow \quad \psi_n(\phi) = \frac{1}{\sqrt{2\pi}}e^{\mathrm{i}n\phi} \quad \text{with} \quad \epsilon_n = \frac{\hbar^2}{2m}\left(n - \frac{\Phi}{\Phi_0}\right)^2. \quad (2.1.1)$$

After the insertion of a full flux quantum $\Phi_0 = h/e$, the Hamiltonian returns to itself. However, if we follow each state adiabatically, we see that the first excited and the ground state exchanged their positions. This situation is called spectral flow: While the spectrum has to be the same for $\Phi = 0$ and $\Phi = \Phi_0$, the adiabatic evolution does not need to return the ground state to itself! This is illustrated in Fig. 2.1. While the example of a particle on a ring is particularly simple, the same situation can occur for a general setup where after the insertion of a flux Φ_0

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[1] The particle only lives on a ring, so we use polar coordinates and forget about the radial part:

$$ H = \frac{1}{2m}\left(-\mathrm{i}\frac{\hbar\partial_\phi}{r} + eA_\phi\right)^2. $$

We write the gauge field as

$$ A_\phi(r) = \begin{cases} \frac{Br}{2} & r < r_c, \\ \frac{Br_c^2}{2r} = \frac{\Phi}{2\pi r} = \frac{\hbar}{er}\frac{\Phi}{\Phi_0} & r \geq r_c. \end{cases} $$

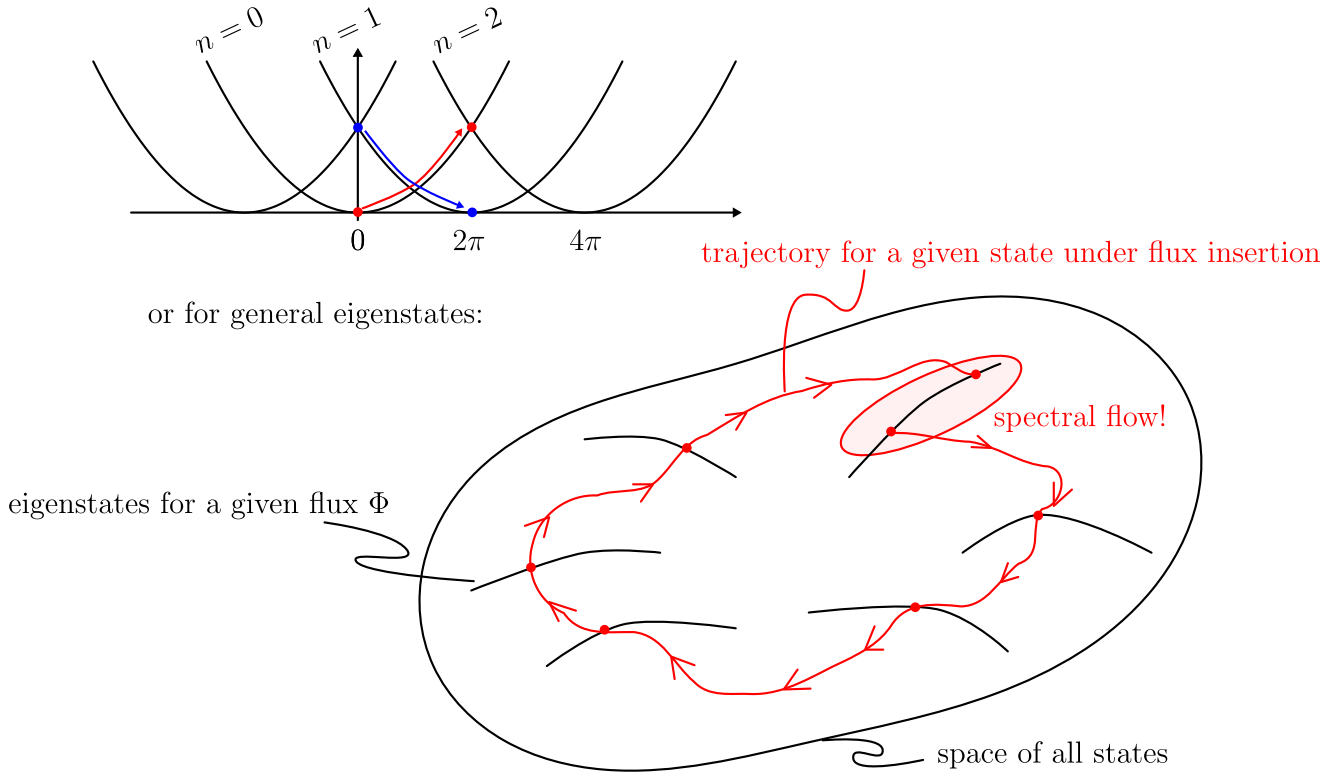At $r = 1$, we find the expression in (2.1.1).

Figure 2.1: Spectral flow.

the original ground state is adiabatically transferred to an excited state. Let us now see how this spectral flow effect applies to the quantum Hall problem.

2.1.2 The ribbon geometry

Laughlin proposed that if σ_{xy} is quantized, it should not depend on the details of the geometry. One is therefore allowed to smoothly deform a rectangular sample as shown in Fig. 2.2. In the last step we replaced the applied voltage $V \to \partial_t \Phi$ with the electromotive force of a time-dependent flux through the opening of the "Corbino" disk.

Let us see what happens when we insert this flux. We make use of the eigenfunctions in the radial gauge $\psi \sim z^m \exp(-z^*z/4)$, where $z = (x - iy)/l$ which we can also write as $e^{-im\phi} r^m \exp(-r^2/4l^2)$. Again, we see that these are Gaussians in one of the coordinate, however, shifted in radial direction depending on m. By calculating $\partial_r \psi = 0$ we find that they are localized around $r_m = \sqrt{2m}l$. Therefore, the flux enclosed by the m'th wave function is given by

$$\pi r_m^2 B = 2\pi m \frac{\hbar}{eB} B = m\Phi_0. \tag{2.1.2}$$

We now add slowly another flux Φ_0 into the opening of the Corbino disk. Slowly means on a time-scale $t_0 \gg 1/\omega_c$, such that we do not excite any particles into the next LL. From the above considerations we conclude that

$$r_m(\Phi) \to r_m(\Phi + \Phi_0) = r_{m+1}(\Phi). \tag{2.1.3}$$

In other words, by inserting a flux quantum we transferred one state from the inner edge of the disk to the outer perimeter. To reach equilibrium, the system will let the charge relax again and we obtain

$$V_{\hat{\phi}} = -\frac{\partial \Phi}{\partial t} = \frac{h}{et_0}; \qquad I_{\hat{r}} = \frac{e}{t_0} \qquad \Rightarrow \qquad \sigma_{xy} = \frac{I_{\hat{r}}}{V_{\hat{\phi}}} = -\frac{e^2}{h}. \tag{2.1.4}$$

This closes the argument of R. Laughlin [1]: The insertion of one flux quantum transfers a quantized charge across the ribbon and hence leads to the quantized Hall conductance measured in the experiment. At this point it is in place to review the assumptions that went into this argument

1. $\hbar/t_0 \ll \hbar\omega_c$, i.e., we adiabatically inserted the flux. This is well justified as σ_{xy} describes *linear response*.

2. Spectral flow lead us to an excited state, i.e., the system was sensitive to the flux insertion!

This last statement is somewhat at odds with the comments in the last chapter on the role of disorder: If we need disorder in an essential way to explain the quantization of σ_{xy}, we also have to accept that disorder in two spatial dimensions leads generically to exponentially localized wave-functions [2], which in turn is incompatible with the sensitivity to flux insertion: how can a localized state in the bulk or in the vicinity of the outer edge feel anything of the flux inserted in the middle?

To reconcile this, one has to understand that a basic assumption of the gang-of-four analysis [2], one-parameter scaling, is violated in the presence of a strong magnetic field. More importantly, we should provide a simple, i.e. semi-classical, picture of how to embed at least one extended bulk state in a disordered Landau level.
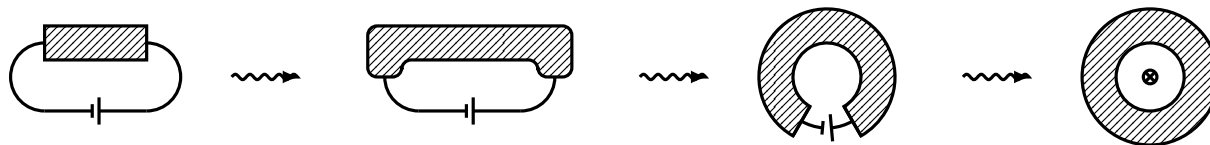

Figure 2.2: Change of geometry for Laughlin's pumping argument.

2.2   The percolation transition

There is a relatively simple picture in terms of percolating clusters that explains the existence of at least one extended state. We know that semi-classical eigenstates in a disordered LL level are given by orbits along equipotential lines.[2] The question is, if there is always an orbit in each LL that connects the two edges of our Corbino disk. If this is the case, this state mediates the sensitivity to the flux insertion and Laughlin's argument goes through also in the disordered case. Luckily the answer to this question is a clear yes!

In Fig. 2.3 an energy landscape for a disordered LLL with a confining potential is shown. Eigenstates are given by equipotential orbits. At low chemical potential μ as shown here, all orbits are "lakes" and hence all states are localized.

When filling in more water (raising μ) we switch at some point from "lakes" to "island". Right at the point where this happens, the shoreline has to connect through the whole sample. This is the sought after extended state in the middle of the sample. Above the center of the LL we are left with "islands" where all states in the bulk are localized. However, we get one edge state on either side of the sample as discussed for the case of no disorder.

We can now summarize our discussion of disorder effects: (i) We found the extended state in the LL needed for Laughlin's pumping argument to hold. (ii) The disorder allows the chemical potential to smoothly change also between the LL's. Therefore, there is an extensive window where the chemical potential lies in the range of the (mobility) gap and hence we find

$$\sigma_{xy} = -\frac{e^2}{h}\nu \qquad \nu \in \mathbb{Z}. \tag{2.2.1}$$
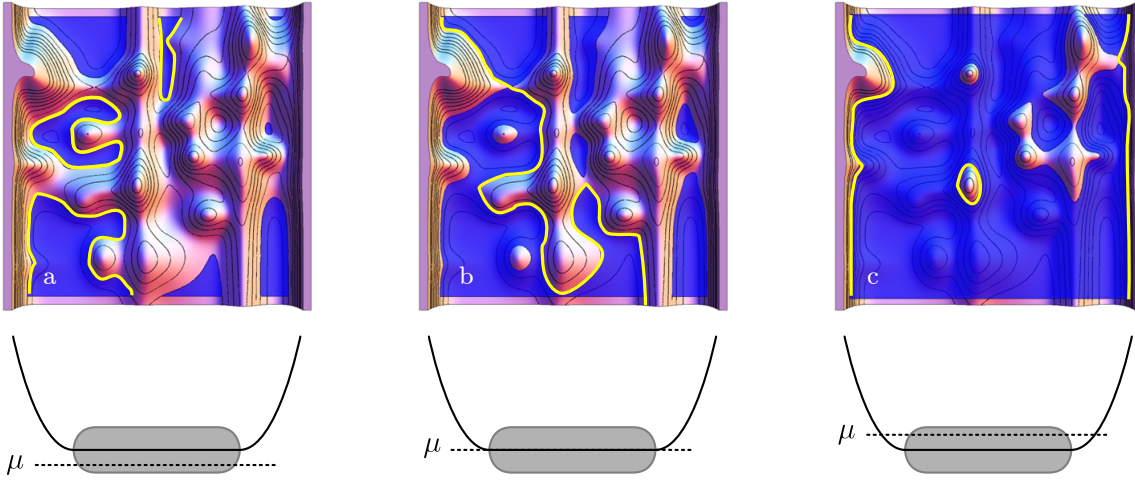
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[2]See exercise number one.

Figure 2.3: (a) Chemical potential below the center of the LL: All states are localized in the form of orbits around lakes. (b) At the percolation threshold there is one shoreline connecting the two sides of the sample. (c) Above the middle of the LL all but the two edge states are localized around islands.

2.3 The TKNN integer

We have now seen that the Hall conductivity has to be quantized in two independent ways. Once, we saw that the edge states of the QHE are chiral one dimensional channels which carry a conductivity of e^2/h. On the other hand, we saw that the pumping argument requires σ_{xy} to be quantized. In both cases the effect was not only stable to disorder but actually required a certain amount of dirt to lift the huge degeneracy of the LL's which made the chemical potential cling to the bulk states. The obvious question is now if there is a deeper, "topological" reason that links these two arguments given above. The answer was given in another seminal paper by Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) in 1982 [3–5].

2.3.1 Landau levels on the torus*

The original paper [3] considered electrons in a periodic potential. Here we want to follow a different route inspired by Avron and Seiler [6] (See also lecture notes by A. Kitaev).
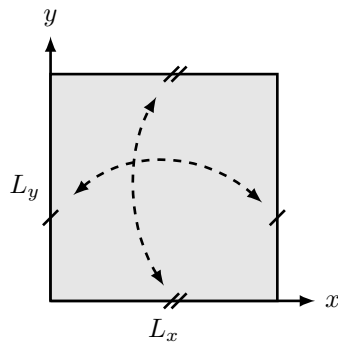

Figure 2.4: Real space torus.

We consider the problem of a magnetic field on a torus. We use the gauge field

$$A_x = \frac{\Phi_x}{L_x}, \quad A_y = \frac{\Phi_y}{L_y} + Bx, \tag{2.3.1}$$

where we added fluxes $\Phi_{x/y}$ through the openings of the torus. The boundary conditions on the torus in the presence of a magnetic field are somewhat non-trivial. Let us define them with respect to the magnetic translation operators which are defined via the canonical momentum operator $(-\mathrm{i}\hbar\nabla - e\boldsymbol{A})$

$$T_{\boldsymbol{u}}^A = e^{\frac{\mathrm{i}}{\hbar}\boldsymbol{u}\cdot\boldsymbol{p}} = e^{\frac{\mathrm{i}}{\hbar}\boldsymbol{u}\cdot(-\mathrm{i}\hbar\nabla - e\boldsymbol{A})}. \tag{2.3.2}$$

Note that these operators depend on the choice of gauge. To derive the boundary conditions,

we now consider $\boldsymbol{u}_1 = (L_x, 0)$ and $\boldsymbol{u}_2 = (0, L_y)$.

$$T^A_{\boldsymbol{u}_1} \psi(x, y) = \psi(x + L_x, y) \overset{!}{=} \psi(x, y), \tag{2.3.3}$$

$$T^A_{\boldsymbol{u}_2} \psi(x, y) = e^{-\frac{ie}{\hbar} B x L_y} \psi(x, y + L_y) \overset{!}{=} \psi(x, y). \tag{2.3.4}$$

These two conditions are only compatible if

$$T^A_{\boldsymbol{u}_1} T^A_{\boldsymbol{u}_2} = T^A_{\boldsymbol{u}_2} T^A_{\boldsymbol{u}_1}. \tag{2.3.5}$$

This is only the case for

$$\frac{eB}{\hbar} L_x L_y = \frac{L_x L_y}{l^2} = 2\pi n \tag{2.3.6}$$

with $n \in \mathbb{Z}$. In other words, an integer number of flux quanta has to pierce the surface of the torus (we can only put quantized magnetic monopoles inside the torus). One can also see that the boundary conditions contain a "gluing phase"

$$\psi(0, y) = \psi(L_x, y), \tag{2.3.7}$$

$$\psi(x, 0) = e^{-\frac{ie}{\hbar} B x L_y} \psi(x, L_y). \tag{2.3.8}$$

In order to appreciate the role of (Φ_x, Φ_y) we calculate the Wilson loops

$$W_x(y) = \oint dx \, \tilde{A}_x(x, y) = BL_x y + \Phi_x, \tag{2.3.9}$$

$$W_y(x) = \oint dy \, \tilde{A}_y(x, y) = BL_y x + \Phi_y, \tag{2.3.10}$$

where we absorbed the gluing phase in the vector potential $\tilde{\boldsymbol{A}}$. $\boldsymbol{W} = (W_x(y), W_y(x))$ is a gauge invariant vector and shows that on a torus a magnetic field breaks all translational symmetries. Moreover, we see, that we can view (Φ_x, Φ_y) as a shift in (x, y). Equipped with these details about the problem of a magnetic field on a torus we now want to embark on the calculation of the Hall conductivity.

2.3.2   Kubo formula

For a microscopic calculation of the conductivity we need a bit of linear response theory. We are interested in the (linear) response of a system to a (small) applied perturbation. In our case the response of interest is the current density $\boldsymbol{j}(r) = \frac{e}{2m} \sum_i [\boldsymbol{p}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) + \delta(\boldsymbol{r} - \boldsymbol{r}_i) \boldsymbol{p}_i]$. The perturbation is given by an applied electric field $\boldsymbol{E} = -\partial_t \boldsymbol{A}$. The perturbing Hamiltonian can therefore be written as

$$H' = -\int d\boldsymbol{r} \, \boldsymbol{j}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}), \tag{2.3.11}$$

We are interested in the expectation value of the current density operator

$$\bar{\boldsymbol{j}}(\boldsymbol{r}, t) = \langle \psi | U^\dagger(t) \boldsymbol{j}(\boldsymbol{r}, t) U(t) | \psi \rangle \qquad \text{with} \qquad U(t) = T_t e^{-\frac{i}{\hbar} \int_{-\infty}^t dt' H'(t')}, \tag{2.3.12}$$

and

$$\boldsymbol{j}(\boldsymbol{r}, t) = e^{\frac{i}{\hbar} \int_{-\infty}^t dt' \, H_0(t')} \boldsymbol{j}(\boldsymbol{r}) e^{-\frac{i}{\hbar} \int_{-\infty}^t dt' \, H_0(t')} \quad \text{and} \quad H'(t) = e^{\frac{i}{\hbar} \int_{-\infty}^t dt' \, H_0(t')} H' e^{-\frac{i}{\hbar} \int_{-\infty}^t dt' \, H_0(t')} \tag{2.3.13}$$

Here, T_t is the time-ordering operator and $|\psi\rangle$ is the unperturbed ground state of the original Hamiltonian H_0. As usual for perturbation theory, we switched to the interaction representation. We assume the vector potential in H' to be given by $\boldsymbol{A} = (\Phi_x/L_x, \Phi_y/L_y) e^{t/\tau}$, which corresponds to slowly turning on the fluxes through the opening of the torus. Moreover, we only drive with

a spatially constant field. Note, that $A_0 = Bx\hat{y}$ is not included in H' as this is not considered to be small but part of the unperturbed Hamiltonian H_0. In linear response we can expand the exponent in $U(t)$ to obtain

$$\bar{j}_\alpha(\boldsymbol{r}, t) = \frac{\mathrm{i}}{\hbar} \sum_\beta \int_{-\infty}^t dt' \, A_\beta(t') \langle [j_\alpha(\boldsymbol{r}, t), \int d\boldsymbol{r}' j_\beta(\boldsymbol{r}', t')] \rangle \tag{2.3.14}$$

$$= \frac{\mathrm{i}}{\hbar} \sum_\beta \int_{-\infty}^t dt' \, A_\beta(t') \langle [j_\alpha(\boldsymbol{r}, t), J_\beta(t')] \rangle. \tag{2.3.15}$$

We write $J_\beta(t')$ for the $\boldsymbol{q} = 0$ Fourier-component of the current as it represent the *total* current. Moreover, as we only drive with $\boldsymbol{q} = 0$, we only get response at this wave vector. To make this clearer we take the Fourier transform on both side with respect to $\boldsymbol{r}$

$$\bar{J}_\alpha(t) = \frac{\mathrm{i}}{\hbar} \sum_\beta \int_{-\infty}^t dt' \, \underbrace{\frac{\Phi_\beta}{L_\beta} e^{\frac{t'}{\tau}}}_{= \tau E_\beta^0 e^{t'/\tau}} \langle [J_\alpha(t), J_\beta(t')] \rangle. \tag{2.3.16}$$

We can now relate the current density[3] to the final driving field E_y^0 to obtain an expression for the Hall conductance

$$\sigma_{xy}(t) = \frac{\mathrm{i}\tau}{\hbar v} \int_{-\infty}^t dt' \, e^{\frac{t'}{\tau}} \langle [J_x(t), J_y(t')] \rangle \tag{2.3.17}$$

The result (2.3.17) is known as the *Kubo formula*. Let us review this result again: The first current operator arises as we measure a current. The second one because the perturbing Hamiltonian H' is also proportional to the current. The commutator originates from the perturbation theory where $U(t)$ is once acting from the left and once from the right. The multiplication by τ accounts for the time derivative linking the electric field $\boldsymbol{E}$ and the vector potential $\boldsymbol{A}$. Finally, in our derivation we made the explicit assumption that we turn on the fluxes Φ_α adiabatically. Certainly the Kubo formula is more general and can be derived for an arbitrary time and space dependence of the perturbation.

To make progress we manipulate (2.3.17) further

$$\sigma_{xy} := \sigma_{xy}(t = 0) = \frac{\mathrm{i}}{\hbar v} \int_{-\infty}^0 dt' \, \tau e^{t'/\tau} \langle [J_\alpha(0), J_\beta(t')] \rangle \tag{2.3.18}$$

$$= \frac{\mathrm{i}}{\hbar v} \int_0^\infty dt_1 \int_{-\infty}^0 dt_2 \langle [J_x(t_1), J_y(t_2)] \rangle e^{-\frac{t_1 - t_2}{\tau}} \tag{2.3.19}$$

$$= \frac{\mathrm{i}}{\hbar} \langle [Q_x^+, Q_y^-] \rangle, \tag{2.3.20}$$

where the operators $Q_\alpha^\pm$ are defined as

$$Q_\alpha^+ = \frac{1}{L_\alpha} \int_0^\infty dt \, e^{-t/\tau} J_\alpha(t), \qquad \text{and} \qquad Q_\alpha^- = \frac{1}{L_\alpha} \int_{-\infty}^0 dt \, e^{t/\tau} J_\alpha(t). \tag{2.3.21}$$

To evaluate the above formula for σ_{xy} we apply the *adiabatic approximation*: We try to exchange the current operators in (2.3.21) with something that explicitly only depends on the *ground state*

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[3]Note that by taking the Fourier-transform

$$\bar{J}(\boldsymbol{q}) = \int d\boldsymbol{r} e^{\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}} \bar{j}(\boldsymbol{r})$$

we switched from current *density* to the total current. However, σ_{xy} relates the driving field $\boldsymbol{E}$ to the *current density* in $\bar{J}_x/v = \bar{J}_x/L_x L_y$. We account for that by dividing by the volume $v = L_x L_y$.

wave-function $|\psi_t\rangle$ at a given time during the turn-on process. At this point it is convenient to introduce the (dimensionless) phases

$$\varphi_\alpha := 2\pi \frac{\Phi_\alpha}{\Phi_0}. \tag{2.3.22}$$

Inserted into (2.3.11) we have

$$H'(t) = -\int d\boldsymbol{r}\, \boldsymbol{j}(\boldsymbol{r},t) \cdot \boldsymbol{A}(\boldsymbol{r},t) = -\sum_\alpha J_\alpha(t) \frac{\Phi_\alpha}{L_\alpha} e^{t/\tau} = -\sum_\alpha J_\alpha(t) \frac{\hbar}{e} \frac{\varphi_\alpha}{L_\alpha} e^{t/\tau}. \tag{2.3.23}$$

Let us write the ground state wave function at $t=0$ by evolving the $t=-\infty$ wave function assuming φ_α to be small

$$|\psi_0\rangle = \hat{T}_t e^{-\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{0} dt'\, H'(t')} |\psi_{-\infty}\rangle \approx |\psi_{-\infty}\rangle + \sum_\alpha \frac{\mathrm{i}}{e} \varphi_\alpha Q_\alpha^- |\psi_{-\infty}\rangle. \tag{2.3.24}$$

We now make use of the adiabatic approximation: We assume that we do not induce any transitions to states above *a postulated energy gap*. Moreover, the state at $t=-\infty$ does not depend on φ_α. When taking the derivative $\partial/\partial\varphi_\alpha$ on both sides of Eq. (2.3.24) we obtain

$$Q_\alpha^- |\psi_{-\infty}\rangle = -\mathrm{i}e \left| \frac{\partial \psi_0}{\partial \varphi_\alpha} \right\rangle. \tag{2.3.25}$$

We achieved our goal to replace the unwanted current operators! For clarity and to make connection to more mathematical literature [7] we introduce the Berry connection

$$\mathcal{A}_\alpha = \mathrm{i}\langle \psi_0 | \partial_\alpha \psi_0 \rangle, \qquad \text{or} \qquad \boldsymbol{\mathcal{A}} = \mathrm{i}\langle \psi_0 | \nabla \psi_0 \rangle \tag{2.3.26}$$

and the corresponding Berry curvature

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha \qquad \text{or} \qquad \boldsymbol{\mathcal{F}} = \nabla \wedge \boldsymbol{\mathcal{A}}. \tag{2.3.27}$$

We can now write for σ_{xy}

$$\sigma_{xy} = \frac{e^2}{\hbar} \mathcal{F}_{xy}, \tag{2.3.28}$$

Let us take the step from the adiabatic turning-on of the field to a dc-field. In that case the field $\boldsymbol{A}$ grows linearly in time, or in other words, the phase φ_x winds as a function of time. We therefore average the above result over $\int d\varphi_x/2\pi$. To make matters more symmetric, we also average over φ_y.[4] This leads us to the formula

$$\sigma_{xy} = \frac{e^2}{\hbar} \int \frac{d\varphi_x d\varphi_y}{(2\pi)^2} \mathcal{F}_{xy} = \frac{e^2}{h} \int \frac{d\varphi_x d\varphi_y}{2\pi} \mathcal{F}_{xy} = \frac{e^2}{h} \frac{\mathsf{C}^{(1)}}{2\pi}, \tag{2.3.29}$$

where we identified the Chern number $\mathsf{C}^{(1)}$ [7]. In order to get a better understanding of Eq. (2.3.29) we related it to the Berry phase of a spin-1/2 in a magnetic field before we motivate it to be quantized to an integer number times the quantum of conductance e^2/h.

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[4]You can argue, that the Hall response should not depend on the arbitrary (but constant) flux through the other opening of the torus. This flux averaging has been widely accepted and recently been put on solid grounds [8].

2.4   The Berry phase

We would like to establish the link between the well known Berry phase [4] and the expression for the Hall conductance derived above. Let us assume that we have a Hamiltonian $H[\boldsymbol{R}(t)]$ that depends on time dependent parameters $\boldsymbol{R}(t)$. These parameters are supposed to evolve slowly

$$\hbar\frac{\partial R_i(t)}{\partial t} \ll \Delta, \tag{2.4.1}$$

where Δ is the minimal gap between the instantaneous ground state and the first excited state at any given time t. If we start at $t = 0$ in the ground state, we will always stay in the instantaneous ground state. However, along the way we will pick up a phase

$$e^{i\varphi(t)}|\psi_0(\boldsymbol{R})\rangle : \qquad i\hbar\partial_t e^{i\varphi(t)}|\psi_0(\boldsymbol{R})\rangle = H[\boldsymbol{R}(t)]e^{i\varphi(t)}|\psi_0(\boldsymbol{R})\rangle. \tag{2.4.2}$$

Multiplying this expression from the left with $\langle\psi_0(\boldsymbol{R})|e^{-i\varphi(t)}$ we obtain

$$\partial_t\varphi(t) = i\langle\psi_0(\boldsymbol{R})|\nabla_{\boldsymbol{R}}|\psi_0(\boldsymbol{R})\rangle \cdot \frac{\partial\boldsymbol{R}}{\partial t} - \frac{1}{\hbar}E_0(\boldsymbol{R}). \tag{2.4.3}$$

Integrating this equation leads to

$$\varphi(t) - \varphi(0) = \underbrace{\int_{\boldsymbol{R}(0)}^{\boldsymbol{R}(t)}\langle\psi_0(\boldsymbol{R})|i\nabla_{\boldsymbol{R}}|\psi_0(\boldsymbol{R})\rangle \cdot d\boldsymbol{R}}_{\text{geometrical phase}} - \underbrace{\frac{1}{\hbar}\int_0^t dt'\,E_0(t')}_{\text{dynamical phase}}\,. \tag{2.4.4}$$

If we now consider a path along a closed contour γ, the dynamical phase drops out and we find

$$\varphi_\gamma = \oint_\gamma d\boldsymbol{l}\,i\langle\psi_0|\nabla_{\boldsymbol{R}}\psi_0\rangle = \oint_\gamma d\boldsymbol{l}\,\boldsymbol{\mathcal{A}} = \int_\Gamma d\boldsymbol{S}\,\boldsymbol{\mathcal{F}}, \tag{2.4.5}$$

where Γ is the area enclosed by the contour γ. With this we see, that σ_{xy} is given by the Berry phase[5] of the ground state when we move the system once around the torus $[0, 2\pi] \times [0, 2\pi]$. Let us gain a deeper understanding of the Berry phase by recalling the example of a spin-1/2 in a magnetic field.

2.4.1   Spin-1/2 in a magnetic field

The Hamiltonian of a spin-1/2 in a magnetic field is given by

$$H = -\boldsymbol{h} \cdot \boldsymbol{\sigma} = -\sum_{\alpha=x,y,z} h_\alpha\sigma_\alpha. \tag{2.4.6}$$

If we write the magnetic field in spherical coordinates

$$h_x = h\sin(\vartheta)\cos(\varphi), \tag{2.4.7}$$
$$h_y = h\sin(\vartheta)\sin(\varphi), \tag{2.4.8}$$
$$h_z = h\cos(\vartheta), \tag{2.4.9}$$

the ground state can be written as

$$|\xi_1\rangle = \begin{pmatrix} \cos(\vartheta/2) \\ e^{i\varphi}\sin(\vartheta/2) \end{pmatrix} \qquad \text{or} \qquad |\xi_2\rangle = \begin{pmatrix} e^{-i\varphi}\cos(\vartheta/2) \\ \sin(\vartheta/2) \end{pmatrix}. \tag{2.4.10}$$

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[5]Why do we call this a phase? Note that

$$0 = \partial_\eta 1 = \partial_\eta\langle\psi|\psi\rangle = \langle\partial_\eta\psi|\psi\rangle + \langle\psi|\partial_\eta\psi\rangle = \langle\partial_\eta\psi|\psi\rangle + \langle\partial_\eta\psi|\psi\rangle^* = a + a^*.$$

Therefore $a = -a^*$ and hence, $\text{Re}[a] = 0$.

Both states $|\xi_1\rangle$ and $|\xi_2\rangle$ describe the ground state. However, $|\xi_1\rangle$ is singular at $\vartheta = \pi$ (or at the south-pole), while $|\xi_2\rangle$ is singular at the north-pole. In other words, we had to introduce two patches on the sphere to obtain a smooth parameterization of the instantaneous eigenstates, see Fig. 2.5. However, we can glue these two patches together via a gluing phase

$$|\xi_1\rangle = e^{i\zeta(\varphi)}|\xi_2\rangle \qquad \text{with} \qquad \zeta(\varphi) = \varphi \tag{2.4.11}$$

along the equator. Let us calculate the Berry connection for the two states. Recall that in spherical coordinates the differential operators take the forms

$$\nabla f = \frac{\partial}{\partial r} f \,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial \vartheta} f \,\hat{\boldsymbol{\vartheta}} + \frac{1}{r \sin(\vartheta)}\frac{\partial}{\partial \varphi} f \,\hat{\boldsymbol{\varphi}}, \tag{2.4.12}$$

and

$$\nabla \wedge \boldsymbol{A} = \frac{1}{r\sin(\vartheta)}\left\{ \frac{\partial}{\partial \vartheta}[A_\varphi \sin(\vartheta)] - \frac{\partial A_\vartheta}{\partial \varphi} \right\}\hat{\boldsymbol{r}} + \frac{1}{r}\left\{ \frac{1}{\sin(\vartheta)}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi) \right\}\hat{\boldsymbol{\vartheta}}$$
$$+ \frac{1}{r}\left\{ \frac{\partial}{\partial r}(rA_\vartheta) - \frac{\partial A_r}{\partial \vartheta} \right\}\hat{\boldsymbol{\varphi}}. \tag{2.4.13}$$

With this we immediately find

$$\boldsymbol{A} = \frac{1}{2r}\hat{\boldsymbol{\varphi}} \cdot \begin{cases} -\tan(\vartheta/2) & |\xi_1\rangle \\ \cot(\vartheta/2) & |\xi_2\rangle \end{cases} \qquad \text{and} \qquad \boldsymbol{\mathcal{F}} = -\frac{\alpha}{2r^2}\hat{\boldsymbol{r}}. \tag{2.4.14}$$

Here, we introduced $\alpha = 1$ for later purposes. A few remarks are in order: (i) The "$\boldsymbol{B}$"-field $\mathcal{F}_{\varphi\vartheta}$ corresponds to a monopole field of a monopole of strength $-\alpha$ at the origin. This can be seen by integrating $\mathcal{F}_{\varphi\vartheta}$ over the whole sphere S^2

$$\int_{S^2} d\boldsymbol{\Omega}\,\boldsymbol{\mathcal{F}} = -\alpha. \tag{2.4.15}$$

(ii) Integrated over the solid angle $d\boldsymbol{\Omega}$ we obtain a Berry phase $\varphi = -\frac{\alpha}{2}d\Omega$. (iii) The field $\boldsymbol{A}$ corresponds to a monopole at $\boldsymbol{h} = 0$. At $\boldsymbol{h} = 0$ the Hamiltonian is zero, i.e., the system is *doubly degenerate*. To appreciate this further, we write (see Ref. [4] for details)

$$\boldsymbol{\mathcal{F}} = \boldsymbol{B} = \nabla \wedge \langle\psi|\nabla\psi\rangle = \langle\nabla\psi| \wedge |\nabla\psi\rangle = \sum_{m \neq 0}\langle\nabla\psi|m\rangle \wedge \langle m|\nabla\psi\rangle = \sum_{m \neq 0}\frac{\langle\psi|\nabla H|m\rangle \wedge \langle m|\nabla H|\psi\rangle}{(E_m - E_0)^2}.$$

It is easy to show that $\nabla \cdot \boldsymbol{B} = 0$ if $E_m - E_0 \neq 0$.

To take another step towards understanding the quantization of the Chern number, let us show that α cannot take arbitrary values. We calculate the Berry phase along a path γ that does not contain the south pole. For simplicity, let us take the equator. We can therefore write

$$\varphi_\gamma = \oint_\gamma d\boldsymbol{l}\,\boldsymbol{A} = \int_\Gamma d\boldsymbol{\Omega}\,\boldsymbol{\mathcal{F}} = -\alpha\Omega(\Gamma)/2 \quad \mathrm{mod}\ 2\pi, \tag{2.4.16}$$

where $\Omega(\Gamma)$ is the solid angle of the surface Γ. We closed γ such that Γ contains the north pole. Alternatively we could have closed γ to $\Gamma' = S^2 - \Gamma$ and write

$$\varphi_\gamma = -\int_{\Gamma'} d\boldsymbol{\Omega}\,\boldsymbol{\mathcal{F}} = \alpha(4\pi - \Omega(\Gamma))/2 \quad \mathrm{mod}\ 2\pi. \tag{2.4.17}$$

In order for (2.4.16) and (2.4.17) to yield the same result we require

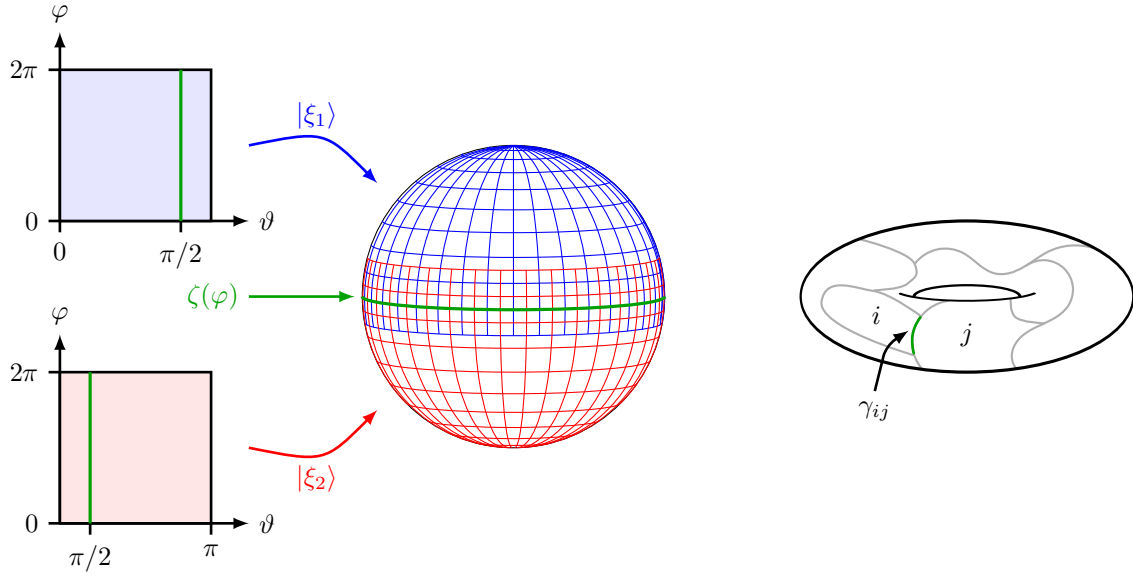$$\alpha \in \mathbb{Z}. \tag{2.4.18}$$

Figure 2.5: Left: Two patches introduced by the gauge choice for the ground state of a spin-1/2 in a magnetic field. No single patch can describe all eigenstate. However, the two patches can be glued together via $\zeta(\varphi)$. Right: Triangulation of the torus where the gluing phase between two individual patches i and j is indicated.

This observation leads us back to the definition of the Chern number

$$\mathsf{C}^{(1)} = \frac{1}{2\pi} \int_M d\mathbf{\Omega}\,\mathcal{F} \quad \in \quad \mathbb{Z} \tag{2.4.19}$$

for any "well behaved", compact, two-dimensional manifold M without boundaries. It can be shown that $\mathsf{C}^{(1)}$ is a topological invariant of the *fiber bundle* looking locally like $M \times U(1)$, where $M = S^2$ is the base manifold defined by the parameters of the Hamiltonian (or more precisely, the parameters defining the projectors $|\psi_0\rangle\langle\psi_0|$ onto the ground state) and $U(1)$ is called the fibre defined by the phase of the ground state at any given point on S^2. Note, however, that the fibre bundle we are dealing with only *locally* looks like $S^2 \times U(1)$. To get the full fibre bundle we need to stitch together the two patches defined by $|\xi_{1/2}\rangle$. The role of the gluing phases ζ_{ij} for patches i, j on the compact manifold M can be further highlighted through the formula

$$\mathsf{C}^{(1)} = \frac{1}{2\pi} \sum_{i<j} \int_{\gamma_{ij}} d\zeta_{ij} \quad \in \quad \mathbb{Z}. \tag{2.4.20}$$

In other words, a non-vanishing Chern number $\mathcal{C}$ is intrinsically linked to the inability to chose a smooth gauge, i.e., only if we have to chose several patches that we glue together with ζ_{ij} can $\mathsf{C}^{(1)}$ be non-zero, see Fig. 2.5. A concrete example of this is our example of the spin-1/2 for which we have

$$\mathsf{C}^{(1)} = \frac{1}{2\pi} \int_{S^2} d\mathbf{\Omega}\,\mathcal{F} = -1 = \frac{1}{2\pi} \oint_{\text{equator}} d\varphi\,\zeta(\varphi). \tag{2.4.21}$$

For further details we refer to the book by Nakahara [7] for a detailed mathematical exposition or the book by Bohm et al. [9] for a more physical approach. We finish this section by stating the long-sought formula

$$\boxed{\sigma_{xy} = \frac{e^2}{h}\nu \qquad \text{with} \qquad \nu \in \mathbb{Z}.} \tag{2.4.22}$$

2.5   Translation invariant systems

Above we made an effort to formulate the derivation of σ_{xy} free of relations to momentum integrals. This allows our formalism to be applied to disordered or interacting systems [6]. However, much of what will follow in Chern insulators and eventually the so-called "topological insulators" will be formulated in translation-invariant systems.

Both for a systems in free space as well as on a lattice it is easy to see that the (quasi) momentum $\boldsymbol{k}$ is doing nothing but making the wave-function acquire a phase $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$ which is linearly growing in $\boldsymbol{x}$. Moreover k_α is the proportionality constant in α-direction. The fluxes (Φ_x, Φ_y) do exactly the same. This can easily be seen by performing a gauge transformation

$$\psi(x) = e^{\mathrm{i}kx} \longrightarrow \psi'(x) = e^{\mathrm{i}\frac{e}{\hbar}\int_0^x dx' A(x')}e^{\mathrm{i}kx} = e^{\mathrm{i}\frac{e}{\hbar}\Phi_x x}e^{\mathrm{i}kx} = e^{\mathrm{i}(k+\varphi)x} = e^{\mathrm{i}k'x}. \tag{2.5.1}$$

We see that therefore the integrals over φ in (2.3.29) are nothing but momentum space integrals for periodic systems [3]. Show to yourself that for a ground state wave-function which is a Slater determinant of momentum eigenstates, the expression (2.3.29) is particularly simple.

2.6   Berry curvature as a magnetic field in momentum space

Before we move on to examples beyond Landau levels which carry a Chern number, we want to get another intuition of what a non-zero Berry curvature represents. We consider electrons in a periodic potential under a weak perturbation. Under the right circumstances one can describe the dynamics in a semiclassical model described by the equations of motion for a wave-packet

$$\dot{\boldsymbol{r}} = \boldsymbol{v}_n(\boldsymbol{k}) = \frac{1}{\hbar}\nabla_{\boldsymbol{k}}\epsilon_n(\boldsymbol{k}), \tag{2.6.1}$$

$$\hbar\dot{\boldsymbol{k}} = -e\left[\boldsymbol{E}(\boldsymbol{r}, t) + \dot{\boldsymbol{r}} \wedge \boldsymbol{B}(\boldsymbol{r}, t)\right], \tag{2.6.2}$$

where n labels the n'th Bloch band. A proper derivation of these equations is beyond the scope of this course. We refer the reader to Ashcroft & Mermin [10] for a basic introduction and to the excellent review by Xiao, et al. [11]. When band properties are taken more properly into account, one finds the above equations have to be adjusted to read [11]

$$\dot{\boldsymbol{r}} = \boldsymbol{v}_n(\boldsymbol{k}) = \frac{1}{\hbar}\nabla_{\boldsymbol{k}}\epsilon_n(\boldsymbol{k}) - \dot{\boldsymbol{k}} \wedge \boldsymbol{\Omega}_n(\boldsymbol{k}), \tag{2.6.3}$$

$$\hbar\dot{\boldsymbol{k}} = -e\left[\boldsymbol{E}(\boldsymbol{r}, t) + \dot{\boldsymbol{r}} \wedge \boldsymbol{B}(\boldsymbol{r}, t)\right], \tag{2.6.4}$$

with

$$\boldsymbol{\Omega}_n(\boldsymbol{k}) = \langle\partial_{k_x}\varphi_n(\boldsymbol{k})| \wedge |\partial_{k_y}\varphi_n(\boldsymbol{k})\rangle = \mathcal{F}_n(\boldsymbol{k}) \tag{2.6.5}$$

the Berry curvature of the n'th band and $\varphi_n(\boldsymbol{k})$ the corresponding Bloch eigenfunctions. From these equations we can conclude that the Berry curvature takes the role of a "magnetic field" in $\boldsymbol{k}$-space.

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