Week 5
Lecture Notes:
Topological Condensed Matter Physics

Sebastian Huber and Titus Neupert

Department of Physics, ETH Zürich
Department of Physics, University of Zürich

If you notice mistakes or typos, please report them to michael.denner@physik.uzh.ch or periv@phys.ethz.ch

Chapter 5

One-dimensional Topological Superconductors

Learning goals

- We understand the Bogoliubov-de-Gennes representation of a mean-field superconducting Hamiltonian and its relation to a Majorana fermion representation.
- We know one-dimensional topological superconductors, their topological invariant, boundary modes and topological classification.
- We understand how interactions reduce the topological classification from $\mathbb{Z}$ to $\mathbb{Z}_8$ in one-dimensional topological superconductors.

- A. Kitaev, Phys.-Usp. **44**, 131 (2001)
- L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011)

5.1 Warmup for superconductivity: 0D superconductors

As a smooth start in the world of superconducting Bogoluibov-de-Gennes mean-field Hamiltonians, we consider impurity sites in s and p-wave superconductors and show that these can experience a transition at which the parity of the many-body ground state changes. To this end, we consider the simplest noninteracting model of a single site and a pair of sites, respectively, but work in its many-body Hilbert space.
In superconductors without further symmetries (which applies to chiral p-wave superconductors) the $\mathbb{Z}_2$ topological index of a 1D superconductor denotes the change in fermion parity of the ground-state as a flux π is inserted through a system with periodic boundary conditions. This parity change can even be observed in zero-dimensional models of isolated impurities and provides basic intuition whether and how a 1D chain of scalar impurities has the potential to undergo a topological phase transition. Here, we discuss this minimal model for a parity changing transition for one and two sites populated with spinless fermions and for a single site populated with spinful fermions.

5.1.1 Spinless fermions

A single spinless fermion cannot exhibit superconducting pairing. Irrespective of that, we see that an on-site chemical potential μ can change the fermion parity P of the ground state (i.e., whether there is an odd or an even number of electrons in the ground state), for the latter is given by sgnμ. In the occupation basis ($|0\rangle, |1\rangle$), the Hamiltonian reads

$$H = \begin{pmatrix} 0 & 0 \\ 0 & -\mu \end{pmatrix}. \tag{5.1.1}$$

Any other terms in the Hamiltonian violate the conservation of fermion parity. The ground state is given by $|0\rangle$ and $|1\rangle$ for $\mu < 0$ and $\mu > 0$, respectively, with opposite parity.
The minimal extension to this model that accounts for superconducting pairing includes two sites with spinless fermions. In this case, we can have triplet – but not singlet – superconducting pairing. In the basis ($|0,0\rangle, |1,0\rangle, |0,1\rangle, |1,1\rangle$), the Hamiltonian reads

$$H = \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & -\mu & t & 0 \\ 0 & t & -\mu & 0 \\ \Delta^* & 0 & 0 & -2\mu \end{pmatrix}, \tag{5.1.2}$$

where t is the hopping integral between the two sites. The energies are

$$\varepsilon^{\text{even}} = \pm\sqrt{|\Delta|^2 + \mu^2} - \mu, \qquad \varepsilon^{\text{odd}} = \pm t - \mu. \tag{5.1.3}$$

The system does not conserve the fermion number anymore, but it conserves its *parity*. Whenever $|t| > |\Delta|$ (commonly referred to as the weak pairing phase), we can induce a parity change of the ground state (protected crossing) by changing the chemical potential at

$$\mu^2 = t^2 - |\Delta|^2. \tag{5.1.4}$$

For smaller $|\mu|$, the ground state has odd parity, for larger $|\mu|$, it has even parity. This is in line with the behavior of bound states of two impurities in p-wave superconductors: They exhibit a protected crossing in the bound state spectrum. The presence of this protected crossing implies the existence of sub-gap Shiba states in the energy spectrum: since a protected crossing has to occur upon varying μ, which could be considered as modeling the scalar impurity strength, it must be that sub-gap $E < |\Delta|$ states exist.

5.1.2 Spinful fermions

A single site with a spinful fermion degree of freedom will allow for singlet superconducting pairing Δ. We also apply a Zeeman field B in the direction of the spin-quantization axis. In the basis ($|0,0\rangle, |\uparrow,0\rangle, |0,\downarrow\rangle, |\uparrow,\downarrow\rangle$), the Hamiltonian reads

$$H = \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & -\mu + B & 0 & 0 \\ 0 & 0 & -\mu - B & 0 \\ \Delta^* & 0 & 0 & -2\mu \end{pmatrix}. \tag{5.1.5}$$

The energies are

$$\varepsilon^{\text{even}} = \pm\sqrt{|\Delta|^2 + \mu^2} - \mu, \qquad \varepsilon^{\text{odd}} = \pm B - \mu, \tag{5.1.6}$$

and the eigenstates

$$\begin{aligned} |\text{even}, \pm\rangle &= \frac{1}{N_\pm}\left[\left(\mu \pm \sqrt{|\Delta|^2 + \mu^2}\right)|0,0\rangle + \Delta^*|\uparrow,\downarrow\rangle\right], \\ |\text{odd}, -\rangle &= |0,\downarrow\rangle, \qquad |\text{odd}, +\rangle = |\uparrow, 0\rangle, \end{aligned} \tag{5.1.7}$$

where $N_\pm$ is an appropriate normalization.

We observe a level crossing protected by parity symmetry at

$$B^2 = |\Delta|^2 + \mu^2. \tag{5.1.8}$$

For smaller $|B|$, the ground state has even parity, for larger $|B|$, it has odd parity. This is congruent with the behavior of a ferromagnetic Shiba chain on an s-wave superconductor. This model also indicates that a density impurity cannot induce a subgap bound state deep in an s-wave superconducting gap, because μ does not induce any phase transition in this model for $B = 0$.

5.1.3 Bogoliubov-de-Gennes formulation and Nambu spinors

Writing out Hamiltonian (5.1.5) in second quantization,

$$H = (B-\mu)c_\uparrow^\dagger c_\uparrow + (-B-\mu)c_\downarrow^\dagger c_\downarrow + \Delta c_\uparrow^\dagger c_\downarrow^\dagger + \Delta^* c_\downarrow c_\uparrow \tag{5.1.9}$$

we find that it cannot be written as a noninteracting Bloch Hamiltonian anymore, but it is still quadratic in the fermion operators. For that reason we can write it as

$$H = \frac{1}{2}\Psi^\dagger h \Psi, \qquad h = \begin{pmatrix} B-\mu & 0 & 0 & \Delta \\ 0 & -B-\mu & -\Delta & 0 \\ 0 & -\Delta^* & -B+\mu & 0 \\ \Delta^* & 0 & 0 & B+\mu \end{pmatrix}, \tag{5.1.10}$$

where $\Psi = (c_\uparrow, c_\downarrow, c_\uparrow^\dagger, c_\downarrow^\dagger)^\mathsf{T}$. This description allows us to reduce the problem to the study of a matrix h (the Bogoliubov-de-Gennes, BdG, Hamiltonian), but it introduces a redundancy in this matrix in the form of an always present PHS

$$\mathcal{P} = P\mathcal{K}, \qquad P = \tau_1 \otimes \sigma_0, \qquad \mathcal{P}h\mathcal{P}^{-1} = -h, \tag{5.1.11}$$

where τ_i, $i = 1, 2, 3$ are the Pauli matrices acting on particle-hole space. Notice that $\mathcal{P}^2 = +1$. Diagonalizing the matrix h yields

$$\begin{aligned} a_{-1} &= \frac{1}{N_+}(0, \mu + \sqrt{\mu^2 + |\Delta|^2}, \Delta^*, 0)^\mathsf{T}, \\ a_1 &= \frac{1}{N_-}(-\mu + \sqrt{\mu^2 + |\Delta|^2}, 0, 0, \Delta^*)^\mathsf{T}, \\ a_{-2} &= \frac{1}{N_+}(-\mu - \sqrt{\mu^2 + |\Delta|^2}, 0, 0, \Delta^*)^\mathsf{T}, \\ a_2 &= \frac{1}{N_-}(0, \mu - \sqrt{\mu^2 + |\Delta|^2}, \Delta^*, 0)^\mathsf{T} \end{aligned} \tag{5.1.12}$$

with eigenvalues $E_1 = -E_{-1} = B + \sqrt{|\Delta|^2 + \mu^2}$, $E_2 = -E_{-2} = -B + \sqrt{|\Delta|^2 + \mu^2}$ and the appropriate normalizations N_+ and N_-. Let un consider the superconducting limit where the energies are ordered for $B > 0$ like $E_{-1} \leq E_{-2} < E_2 \leq E_1$. We can now define the operators

$$\begin{aligned} \gamma_{-1} &:= a_{-1}^*\Psi^\dagger = \frac{1}{N_+}\left[\left(\mu + \sqrt{\mu^2 + |\Delta|^2}\right) c_\downarrow^\dagger + \Delta c_\uparrow\right], \\ \gamma_1 &:= a_1^*\Psi^\dagger = \frac{1}{N_-}\left[\left(-\mu + \sqrt{\mu^2 + |\Delta|^2}\right) c_\uparrow^\dagger + \Delta c_\downarrow\right], \\ \gamma_{-2} &:= a_{-2}^*\Psi^\dagger = \frac{1}{N_+}\left[\left(-\mu - \sqrt{\mu^2 + |\Delta|^2}\right) c_\uparrow^\dagger + \Delta c_\downarrow\right], \\ \gamma_2 &:= a_2^*\Psi^\dagger = \frac{1}{N_-}\left[\left(\mu - \sqrt{\mu^2 + |\Delta|^2}\right) c_\downarrow^\dagger + \Delta c_\uparrow\right]. \end{aligned} \tag{5.1.13}$$

We check that

$$\gamma_{-1}|\text{even},-\rangle = 0, \qquad \gamma_{-2}|\text{even},-\rangle = 0, \tag{5.1.14}$$

i.e., the ground state is annihilated by the negative energy operators. This defines the BCS ground state. Furthermore,

$$\gamma_1|\text{even},-\rangle = -|\text{odd},+\rangle, \qquad \gamma_2|\text{even},-\rangle = |\text{odd},-\rangle, \qquad \gamma_1\gamma_2|\text{even},-\rangle \propto |\text{even},+\rangle, \tag{5.1.15}$$

that is, we can reach all excited states by applying the respective positive energy operators to the ground state.
In the limit where B dominates, for $B > 0$ the energies are ordered as $E_{-1} \leq E_2 < E_{-2} \leq E_1$. In this case, the ground state $|\text{odd},-\rangle = |0,\downarrow\rangle$ is annihilated by the negative energy operators γ_{-1} and γ_2 and excited states can be constructed from this ground state similar to the above.
A more general translational invariant system with Hamiltonian

$$H = \sum_{\boldsymbol{k}} \sum_{s,s'=\uparrow,\downarrow} \left[c^\dagger_{s,\boldsymbol{k}} \left(h_{0,\boldsymbol{k}}\right)_{s,s'} c_{s',\boldsymbol{k}} + c^\dagger_{s,\boldsymbol{k}} \left(\Delta_{\boldsymbol{k}}\right)_{s,s'} c^\dagger_{s',-\boldsymbol{k}} + \text{h.c.}\right] \tag{5.1.16}$$

can be recast as

$$H = \sum_{\boldsymbol{k}} \Psi^\dagger_{\boldsymbol{k}} h_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}, \qquad h_{\text{k}} = \begin{pmatrix} h_{0,\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta^\dagger_{\boldsymbol{k}} & -h^*_{0,-\boldsymbol{k}} \end{pmatrix}, \tag{5.1.17}$$

where $\Psi_{\boldsymbol{k}} = (c_{\uparrow,\boldsymbol{k}}, c_{\downarrow,\boldsymbol{k}}, c^\dagger_{\uparrow,-\boldsymbol{k}}, c^\dagger_{\downarrow,-\boldsymbol{k}})^\mathsf{T}$ and $h_{0,\boldsymbol{k}}$ as well as $\Delta_{\boldsymbol{k}}$ are $2{\times}2$ matrices. The BCS ground state is then again defined by the unique state that is annihilated by all operators $\gamma_{-1,\boldsymbol{k}}$ and $\gamma_{-2,\boldsymbol{k}}$ with negative energies and excited states are constructed by applying all positive energy operators to it.

5.2 The one-dimensional p-wave superconductor

In the Su-Schrieffer-Heeger model, particle-hole symmetry (and with it the chiral symmetry) is in some sense fine-tuned, as it is lost if generic longer-range hoppings are considered. In superconductors, particle-hole symmetry arises more naturally as a symmetry that is inherent in the redundant description of mean-field Bogoliubov-de-Gennes Hamiltonians.
Before we consider a simplified microscopic model, let us name a set of possible physical ingredients that would be required to realize such a model. They are

- a quasi-1D electronic system
- Rashba spin-orbit coupling
- Zeeman coupling
- proximity-induced s-wave superconductivity.

When corroborating in the correct way, these ingredients yield a 1D system of effectively spinless electrons that are superconducting. A simple first quantized Hamiltonian for the 1D wire with Rashba spin-orbit coupling α and Zeeman coupling B is given by

$$H_{\text{wire}} = \frac{k^2}{2m}\sigma_0 + \alpha k \sigma_y + B\sigma_x - \mu. \tag{5.2.1}$$

The spectrum $\varepsilon_{k,\pm} = \frac{k^2}{2m} - \mu \pm \sqrt{(\alpha k)^2 + B^2}$ has only two Fermi points $k_\pm$ (instead of 4) if $|B| > |\mu|$ and the spin polarization of the two Fermi points is almost opposite in the limit of small B. This means that the two states at the Fermi points are almost Kramers pairs and hence conventional Cooper pairs can effectively couple to this system, gapping out the Fermi points.

5.2.1 The Kitaev wire

Here, we want to consider the simplest model for a topological superconductor that has been studied by Kitaev. The setup is again a 1D chain with one orbital for spinless fermion on each site (the spinless nature is essentially motivated by the fact that at the Fermi points we find eigenstates of a definite spin polarization). Superconductivity is encoded in pairing terms $c_i^\dagger c_{i+1}^\dagger$ that do not conserve particle number. The Hamiltonian is given by

$$H = \sum_{i=1}^{N} \left[-t \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) - \mu c_i^\dagger c_i + \Delta c_{i+1}^\dagger c_i^\dagger + \Delta^* c_i c_{i+1} \right]. \tag{5.2.2}$$

Here, μ is the chemical potential and Δ is the superconducting order parameter, which we will decompose into its amplitude $|\Delta|$ and complex phase ϑ, i.e., $\Delta = |\Delta| e^{\mathrm{i}\vartheta}$.
The fermionic operators $c_i^\dagger$ obey the algebra

$$\{c_i^\dagger, c_j\} = \delta_{i,j}, \tag{5.2.3}$$

with all other anticommutators vanishing. We can chose to trade the operators $c_i^\dagger$ and c_i on every site i for two other operators a_i and b_i that are defined by

$$a_i = e^{-\mathrm{i}\vartheta/2} c_i + e^{\mathrm{i}\vartheta/2} c_i^\dagger, \qquad b_i = \frac{1}{\mathrm{i}} \left(e^{-\mathrm{i}\vartheta/2} c_i - e^{\mathrm{i}\vartheta/2} c_i^\dagger \right). \tag{5.2.4}$$

These so-called Majorana operators obey the algebra

$$\{a_i, a_j\} = \{b_i, b_j\} = 2\delta_{ij}, \qquad \{a_i, b_j\} = 0 \qquad \forall i, j. \tag{5.2.5}$$

In particular, they square to 1

$$a_i^2 = b_i^2 = 1, \tag{5.2.6}$$

and are self-conjugate

$$a_i^\dagger = a_i, \qquad b_i^\dagger = b_i. \tag{5.2.7}$$

In fact, we can always break up a complex fermion operator on a lattice site into its real and imaginary Majorana components though it may not always be a useful representation. As an aside, note that the Majorana anti-commutation relation in Eq. (5.2.5) is the same as that of the generators of a Clifford algebra where the generators all square to +1. Thus, mathematically one can think of the operators a_i (or b_i) as matrices forming by themselves the representation of Clifford algebra generators.
When rewritten in the Majorana operators, Hamiltonian (5.2.2) takes (up to a constant) the form

$$H = \frac{\mathrm{i}}{2} \sum_{i=1}^{N} \left[-\mu a_i\, b_i + (t + |\Delta|) b_i\, a_{i+1} + (-t + |\Delta|) a_i\, b_{i+1} \right]. \tag{5.2.8}$$

After imposing periodic boundary conditions, it is again convenient to study the system in momentum space. When defining the Fourier transform of the Majorana operators $a_i = \sum_i e^{\mathrm{i}ki} a_k$ we note that the the self-conjugate property (5.2.7) that is local in position space translates into $a_k^\dagger = a_{-k}$ in momentum space (and likewise for the b_k). The momentum space representation of the Hamiltonian is

$$\begin{aligned} H &= \sum_{k \in \mathrm{BZ}} \sum_{\alpha = A,B} (a_k \;\; b_k)\, h_k \begin{pmatrix} a_{-k} \\ b_{-k} \end{pmatrix} && \text{(5.2.9a)} \\ h_k &= \begin{pmatrix} 0 & -\frac{\mathrm{i}\mu}{2} + \mathrm{i}t \cos k + |\Delta| \sin k \\ \frac{\mathrm{i}\mu}{2} - \mathrm{i}t \cos k + |\Delta| \sin k & 0 \end{pmatrix} && \text{(5.2.9b)} \\ &= \sigma_x |\Delta| \sin k + \sigma_y \left(\frac{\mu}{2} - t \cos k \right), && \text{(5.2.9c)} \end{aligned}$$

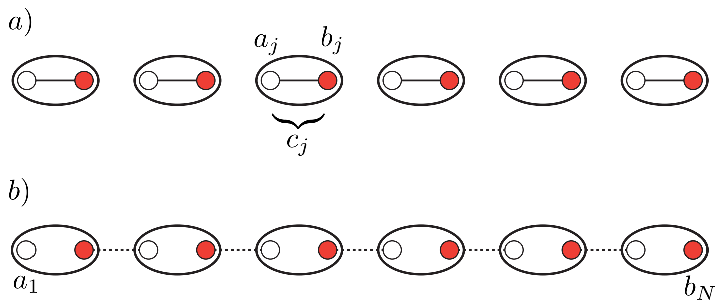

Figure 5.1: Schematic illustration of the lattice p-wave superconductor Hamiltonian in the (a) trivial limit (b) non-trivial limit. The white (empty) and red (filled) circles represent the Majorana fermions making up each physical site (oval). The fermion operator on each physical site (c_j) is split up into two Majorana operators (a_j and b_j). In the non-trivial phase the unpaired Majorana fermion states at the end of the chain are labelled with a_1 and b_N. These are the states which are continuously connected to the zero-modes in the non-trivial topological superconductor phase.

While this Bloch Hamiltonian is formally very similar to that of the SSH model, we have to keep in mind that it acts on entirely different single-particle degrees of freedom, namely in the space of Majorana operators instead of complex fermionic operators. As with the case of the Su-Schrieffer-Heeger model, the Hamiltonian (5.2.9) has a time-reversal symmetry $\mathcal{T} = \sigma_z \mathcal{K}$ and a particle-hole symmetry $P = \mathcal{K}$ which combine to the chiral symmetry $C = \sigma_z$. For the topological properties that we first explore, only the particle-hole symmetry is crucial. We will see that the model has a $\mathbb{Z}_2$ topological classification in this case. We will then take the TRS as a real symmetry in addition, in which case we can again define a winding number, analogous to the SSH model. This results in a $\mathbb{Z}$ topological classification if only Hamiltonians bilinear in the Majorana operators are considered (so-called noninteracting systems).
To determine its topological phases, we notice that Hamiltonian (5.2.9) is gapped except for $|t| = |\mu/2|$. We specialize again on convenient parameter values on either side of this potential topological phase transition

- $\mu = 0,\ |\Delta| = t$: The Bloch matrix h_k takes exactly the same form as that of the SSH model for the parameter choice $\delta = +1$. We conclude that the Hamiltonian (5.2.9) is in a topological phase. The Hamiltonian reduces to

$$H = \mathrm{i}t \sum_j b_j a_{j+1}. \tag{5.2.10}$$

  A pictorial representation of this Hamiltonian is shown in Fig. 5.1 b). With open boundary conditions it is clear that the Majorana operators a_1 and b_N are not coupled to the rest of the chain and are ‘unpaired’. In this limit the existence of two Majorana zero modes localized on the ends of the chain is manifest.

- $\Delta = t = 0,\ \mu < 0$: This is the topologically trivial phase, since the Hamiltonian is independent of k so that that the winding number vanishes necessarily. In this case the Hamiltonian reduces to

$$H \quad = \quad -\mu \frac{\mathrm{i}}{2} \sum_j a_j b_j. \tag{5.2.11}$$

  In its ground state the Majorana operators on each physical site are coupled but the Majorana operators between each physical site are decoupled. In terms of the physical complex fermions, it is the ground state with either all sites occupied or all sites empty. A representation of this Hamiltonian is shown in Fig. 5.1 a). The Hamiltonian in the physical-site basis is in the atomic limit, which is another way to see that the ground state is trivial. If the chain has open boundary conditions there will be no low-energy states on the end of the chain if the boundaries are cut between *physical* sites. That is, we are not allowed to pick boundary conditions where a physical complex fermionic site is cut in half.

These two limits give the simplest representations of the trivial and non-trivial phases. By tuning away from these limits the Hamiltonian will have some mixture of couplings between Majorana operators on the same physical site, and operators between physical sites. However, since the two Majorana modes are localized at different ends of a gapped chain, the coupling between them will be exponentially small in the length of the wire and they will remain at zero energy. In fact, in the non-trivial phase the zero modes will not be destroyed until the bulk gap closes at a critical point.
It is important to note that these zero modes count to a different many-body ground state degeneracy than the end modes of the Su-Schrieffer-Heeger model. The difference is rooted in the fact that one cannot build a fermionic Fock space out of an odd number of Majorana modes, because they are linear combinations of particles and holes. Rather, we can define a *single* fermionic operator out of *both* Majorana end modes a_1 and b_N as $c^\dagger := a_1 + \mathrm{i}b_N$. The Hilbert space we can build out of a_1 and b_N is hence inherently nonlocal. This *nonlocal* state can be either occupied or empty giving rise to a two-fold degenerate ground state of the chain with two open ends. (In contrast, the topological Su-Schrieffer-Heeger chain has a four-fold degenerate ground state with two open ends, because it has one fermionic mode on each end.) The Majorana chain thus displays a different form of fractionalization than the Su-Schrieffer-Heeger chain. For the latter the topological end modes carry fractional charge. In the Majorana chain, the end modes are a fractionalization of a fermionic mode into a superposition of particle and hole (and have no well defined charge anymore), but the states $|0\rangle$ (with $c|0\rangle = 0$) and $c^\dagger|0\rangle$ do have distinct fermion parity. The nonlocal fermionic mode formed by two Majorana end modes is envisioned to work as a qubit (a quantum-mechanical two-level system) that stores quantum information (its state) in a way that is protected against local noise and decoherence.

5.2.2 Topological classification

If we disregard the time-reversal symmetry, and two parallel wires are considered, we can gap out the two end states by a term
$$\mathrm{i}a_1 a_1', \tag{5.2.12}$$
where the primed and unprimed operator are the end states of the first and the second wire, respectively. This implies that, while a single end state is protected, a pair of them is not. The topological classification with only PHS is $\mathbb{Z}_2$. Let us now discuss the topological index for this case. Intuitively, when we consider pairing between opposite momentum eigenstates, the topological invariant should distinguish the two cases where an even number of pairs of Fermi points was present before the pairing was introduced from the situation where an odd number of pairs of Fermi points was present. The former would be topologically trivial, while the latter would be the non-trivial superconductor. The points $k = 0$ and $k = \pi$ are invariant under PHS and it is sufficient to determine the parity of the number of occupied bands at $k = 0$ and $k = \pi$ to deduce the parity of pairs of Fermi points. If the product of the parities of occupied bands is odd, there is an odd number of Fermi points. It is not possible to deduce from the BdG Hamiltonian the parity of the number of occupied bands without particle-hole doubling. In a sense, we would need to take the square root of it in a controlled way and then compute the sign of the square root. From Hamiltonian (5.2.9) we make two observations

1. h_0 and h_π have the form i times an antisymmetric matrix

2. the change of the sign of the upper right (or lower left) element between $k = 0$ and $k = \pi$ is what relates to the change in parity between the two points.

For a general Majorana Hamiltonian,

$$H = \frac{\mathrm{i}}{2} \sum_{r,r'} \gamma_r^{\mathsf{T}} A_{r,r'} \gamma_{r'} \tag{5.2.13}$$

these observations can be generalized by observing that

$$H = \frac{\mathrm{i}}{2} \sum_{k} \tilde{\gamma}_{-k}^{\mathsf{T}} \tilde{A}(k) \tilde{\gamma}_k, \tag{5.2.14}$$

where we have defined

$$\tilde{A}(k) = \sum_R e^{\mathrm{i}kR} A_R \tag{5.2.15}$$

assumed translational invariance $A_{r,r'} = A_{r-r'} = A_R$. Since $A_{r,r'}$ is an antisymmetric matrix, $\tilde{A}(k)$ are also antisymmetric for $k = 0, \pi$. For even dimensional antisymmetric matrices, the Pfaffian

$$\mathrm{Pf}(A) := \frac{1}{2^n n!} \epsilon_{i_1,i_2,\cdots,i_{2n}} A_{i_1,i_2} A_{i_3,i_4} \cdots A_{i_{2n-1},i_{2n}}, \tag{5.2.16}$$

is a way of taking the square root of the determinant. The topological $\mathbb{Z}_2$ invariant is given by

$$\nu = \mathrm{sgn}\left\{\mathrm{Pf}[\tilde{A}(0)]\mathrm{Pf}[\tilde{A}(\pi)]\right\} = \pm 1. \tag{5.2.17}$$

If translational symmetry is not present, this generalizes to

$$\nu = \mathrm{sgn}\,\mathrm{Pf}(A). \tag{5.2.18}$$

If we also insist on the presence of TRS, the chiral symmetry guarantees that the winding number is a well-defined topological invariant similar to the case of the SSH model yielding a $\mathbb{Z}$ topological classification if only bilinear (noninteracting) Hamiltonians are considered. For interacting Hamiltonians, something more interesting happens, as we explore in the next section.

5.2.3 Reduction of the classification by interactions: $\mathbb{Z} \to \mathbb{Z}_8$

When time-reversal symmetry $\mathcal{T} = \mathcal{K}$ is present, the model considered in Sec. 5.2 has a noninteracting $\mathbb{Z}$ topological characterization. We want to explore how interactions alter this classification, following a calculation by Fidkowski and Kitaev. To this end, we consider a collection of n identical 1D topological Majorana chains and only consider their Majorana end modes on one end, which we denote by $a_1, \cdots, a_n$. We will take the point of view that if we can gap the edge, we can continue the bulk to a trivial state (insulator). This is not entirely a correct point of view in general (see 2D topologically ordered states such as the toric code discussed in the next Section), but works for our purposes. Given some integer n, we ask whether we can couple the Majorana modes locally on one end such that no gapless degrees of freedom are left on that end and the ground state with open boundary conditions becomes singly degenerate. We only allow couplings that respect time-reversal symmetry. Let us first derive the action of $\mathcal{T}$ on the Majorana modes. The complex fermion operators are left invariant under time-reversal $\mathcal{T} c \mathcal{T}^{-1} = c$. Hence,

$$\mathcal{T}(a + \mathrm{i}b)\mathcal{T}^{-1} = \mathcal{T} a \mathcal{T}^{-1} - \mathrm{i}\mathcal{T} b \mathcal{T}^{-1} \stackrel{!}{=} a + \mathrm{i}b \quad \Rightarrow \quad \mathcal{T} a \mathcal{T}^{-1} = a, \qquad \mathcal{T} b \mathcal{T}^{-1} = -b. \tag{5.2.19}$$

Thus, when acting on the modes localized on the left end of the wire (which transform like the a's), time-reversal symmetry leaves the Majorana operators invariant.

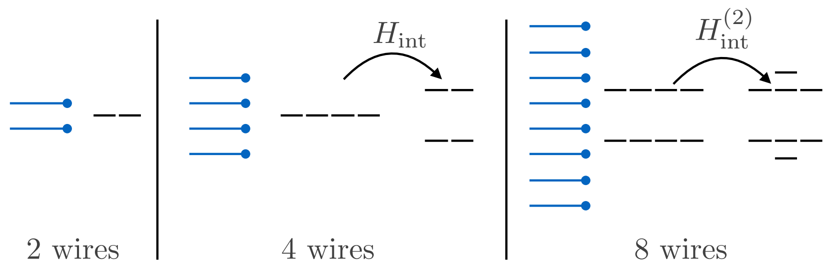

Figure 5.2: Schematic illustration of the many body energy levels for 2, 4, and 8 wires with Majorana end states as well as the (partial) lifting of their degeneracy that is in accordance with time-reversal symmetry.

One now subsequently considers the Majorana end modes from 2,4, and 8 wires and adds suitable perturbations to see whether a unique (many-body) ground state can be obtained while retaining time-reversal symmetry. One finds doubly-degenerate ground state for 2 as well as 4 wires (see exercise). (We note in passing that the ground state from the end states of 4 wires is even 4-fold degenerate if only non-interacting, i.e., Majorana-bilinear terms are allowed.) Going to 8 wires, one can construct an interaction term between them that has a unique ground state (see Fig. 5.2).
This unique ground state can be adiabatically continued to the atomic limit. In this way the noninteracting $\mathbb{Z}$ classification breaks down to $\mathbb{Z}_8$ if interactions are allowed.

References

1. Kitaev, A. Y. "Unpaired Majorana fermions in quantum wires". *Phys.-Usp.* **44,** 131 (2001).
2. Fidkowski, L. "Entanglement Spectrum of Topological Insulators and Superconductors". *Phys. Rev. Lett.* **104,** 130502. `http://link.aps.org/doi/10.1103/PhysRevLett.104.130502` (2010).