Week 6
Lecture Notes:
Topological Condensed Matter Physics

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Chapter 6

Two-dimensional Topological Superconductors

Learning goals

- We know the chiral p-wave superconductor in two dimensions and can argue why it has bound states in vortices.
- We understand the non-Abelian nature of vortex bound states.
- We can motivate Kitaev's 16-fold way classification for 2D superconductors.

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6.1 Lattice and continuum model and their topological invariant

After having studied topological superconductivity in 1D, we now want to move to 2D where we will find qualitatively new physics in the chiral p-wave superconductor. On the level of the noninteracting Bloch Hamiltonian, it is formally similar to a Chern insulator, but we will see that the physical degrees of freedom on which this Hamiltonian acts make the story much richer, bridging to theories with anyonic excitations and topological order. More precisely, the vortices of the chiral p-wave superconductor exhibit anyon excitations which have exotic non-Abelian statistics. For the system to be topologically ordered, these vortices should appear as emergent, dynamical excitations. This requires to treat the electromagnetic gauge field quantum-mechanically. (In fact, since the fermion number conservation is spontaneously broken down to the conservation of the fermion parity in the superconductor, the relevant gauge theory involves only a $\mathbb{Z}_2$ instead of a U(1) gauge field.) However, the topological properties that we want to discuss here can also be seen if we model the gauge field and vortices as static defects, rather than within a fluctuating $\mathbb{Z}_2$ gauge theory. This allows us to study a models very similar to the "noninteracting" topological superconductor in 1D and still expose the non-Abelian statistics. For pedagogy we will use both lattice and continuum models of the chiral superconductor. We begin with the lattice Hamiltonian defined on a square lattice

$$
\begin{aligned}
H = \sum_{m,n} \Big\{ &-t\left(c^\dagger_{m+1,n}c_{m,n} + c^\dagger_{m,n+1}c_{m,n} + \text{h.c.}\right) - (\mu - 4t)c^\dagger_{m,n}c_{m,n} \\
&+ \left(\Delta c^\dagger_{m+1,n}c^\dagger_{m,n} + \mathrm{i}\Delta c^\dagger_{m,n+1}c^\dagger_{m,n} + \text{h.c.}\right)\Big\}.
\end{aligned}
\tag{6.1.1}
$$

The fermion operators $c_{m,n}$ annihilate fermions on the lattice site (m,n) and we are considering spinless (or equivalently spin-polarized) fermions. We set the lattice constant $a = 1$ for simplicity. The pairing amplitude is anisotropic and has an additional phase of i in the y-direction compared to the pairing in the x-direction. Because the pairing is not on-site, just as in the lattice version of the p-wave wire, the pairing terms will have momentum dependence. We can write this Hamiltonian in the Bogoliubov-deGennes form and, assuming that Δ is translationally invariant, can Fourier transform the lattice model to get

$$H_{\mathrm{BdG}} = \frac{1}{2}\sum_{\boldsymbol{p}} \Psi_{\boldsymbol{p}}^\dagger \begin{pmatrix} \epsilon(p) & 2\mathrm{i}\Delta(\sin p_x + \mathrm{i}\sin p_y) \\ -2\mathrm{i}\Delta^*(\sin p_x - \mathrm{i}\sin p_y) & -\epsilon(p) \end{pmatrix} \Psi_{\boldsymbol{p}}, \tag{6.1.2}$$

where $\epsilon(p) = -2t(\cos p_x + \cos p_y) - (\mu - 4t)$ and $\Psi_{\boldsymbol{p}} = \begin{pmatrix} c_{\boldsymbol{p}} & c_{-\boldsymbol{p}}^\dagger \end{pmatrix}^\mathsf{T}$. For convenience we have shifted the chemical potential by the constant $4t$. As a quick aside we note that the model takes a simple familiar form in the continuum limit ($\boldsymbol{p} \to 0$):

$$H_{\mathrm{BdG}}^{(\mathrm{cont})} = \frac{1}{2}\sum_{\boldsymbol{p}} \Psi_{\boldsymbol{p}}^\dagger \begin{pmatrix} \frac{p^2}{2m} - \mu & 2\mathrm{i}\Delta(p_x + \mathrm{i}p_y) \\ -2\mathrm{i}\Delta^*(p_x - \mathrm{i}p_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \Psi_{\boldsymbol{p}} \tag{6.1.3}$$

where $m \equiv 1/2t$ and $p^2 = p_x^2 + p_y^2$. We see that the continuum limit has the characteristic $p_x + \mathrm{i}p_y$ chiral form for the pairing potential. The quasiparticle spectrum of $H_{\mathrm{BdG}}^{(\mathrm{cont})}$ is $E_\pm = \pm\sqrt{(p^2/2m - \mu)^2 + 4|\Delta|^2 p^2}$, which, with a nonvanishing pairing amplitude, is gapped across the entire BZ as long as $\mu \neq 0$. This is unlike some other types of p-wave pairing terms [e.g., $\Delta(p) = \Delta p_x$] which can have gapless *nodal* points or lines in the BZ for $\mu > 0$. In fact, nodal superconductors, having gapless quasiparticle spectra, are not topological superconductors by definition (i.e., a bulk excitation gap does not exist).

We recognize the form of $H_{\mathrm{BdG}}^{(\mathrm{cont})}$ as a massive 2D Dirac Hamiltonian, and indeed Eq. (6.1.1) is just a lattice Dirac Hamiltonian which is what we will consider first. In the first quantized notation, the single particle Hamiltonian for a superconductor is equivalent to that of an insulator with an additional particle-hole symmetry (It is thus placed in class D in the classification that we will introduce in the next lecture) and admits a $\mathbb{Z}$ topological classification in 2D. Thus, we can classify the eigenstates of Hamiltonian (6.1.1) by a Chern number – but due to the breaking of U(1) symmetry, the Chern number does not have the interpretation of Hall conductance. However, it is still a topological invariant.

We expect that H_{BdG} will exhibit several phases as a function of Δ and μ for a fixed $t > 0$. For simplicity let us set $t = 1/2$ and make a gauge transformation $c_{\boldsymbol{p}} \to e^{\mathrm{i}\vartheta/2} c_{\boldsymbol{p}},\ c_{\boldsymbol{p}}^\dagger \to e^{-\mathrm{i}\vartheta/2} c_{\boldsymbol{p}}^\dagger$ where $\Delta = |\Delta| e^{i\vartheta}$. The Bloch Hamiltonian for the lattice superconductor is then

$$\mathcal{H}_{\mathrm{BdG}}(\boldsymbol{p}) = (2 - \mu - \cos p_x - \cos p_y)\,\sigma_z - 2|\Delta| \sin p_x \sigma_y - 2|\Delta| \sin p_y \sigma_x, \tag{6.1.4}$$

where the $\sigma_i,\ i = x, y, z$, are the Pauli matrices in the particle/hole basis. Assuming $|\Delta| \neq 0$, this Hamiltonian has several fully-gapped superconducting phases separated by gapless critical points. The quasi-particle spectrum for the lattice model is

$$E_\pm = \pm\sqrt{(2 - \mu - \cos p_x - \cos p_y)^2 + 4|\Delta|^2 \sin^2 p_x + 4|\Delta|^2 \sin^2 p_y} \tag{6.1.5}$$

and is gapped (under the assumption that $|\Delta| \neq 0$) unless the prefactors of all three Pauli matrices vanish simultaneously. As a function of (p_x, p_y, μ) we find three critical points. The first critical point occurs at $(p_x, p_y, \mu) = (0, 0, 0)$. The second critical point has two gap-closings in the BZ for the same value of μ : $(\pi, 0, 2)$ and $(0, \pi, 2)$. The third critical point is again a singly degenerate point at $(\pi, \pi, 4)$. We will show that the phases for $\mu < 0$ and $\mu > 4$ are trivial superconductors while the phases $0 < \mu < 2$ and $2 < \mu < 4$ are topological superconductors with opposite chirality. In principle one can define a Chern number topological invariant constructed from the eigenstates of the lower quasi-particle band to characterize the phases. We will show this calculation below, but first we make some physical arguments as to the nature of the phases. We will first consider the phase transition at $\mu = 0$. The low-energy physics for this transition occurs around $(p_x, p_y) = (0, 0)$ and so we can expand the lattice Hamiltonian around this point; this is nothing but Eq. (6.1.3). One way to test the character of the $\mu < 0$ and $\mu > 0$ phases is to make an interface between them. If we can find a continuous interpolation between these two regimes which is always gapped then they are topologically equivalent phases of matter. If we cannot find such a continuously gapped interpolation then they are topologically distinct. A simple geometry to study is a domain wall where $\mu = \mu(x)$ such that $\mu(x) = -\mu_0$ for $x < 0$ and $\mu(x) = +\mu_0$ for $x > 0$ for a positive constant μ_0. This is an interface which is translationally invariant along the y-direction, and thus we can consider the momentum p_y as a good quantum number to simplify the calculation. What we will now show is that there exist gapless, propagating fermions bound to the interface which prevent us from continuously connecting the $\mu < 0$ phase to the $\mu > 0$ phase. This is one indication that the two phases represent topologically distinct classes.

The single-particle Hamiltonian in this geometry is

$$\mathcal{H}_{\text{BdG}}(p_y) = \frac{1}{2}\begin{pmatrix} -\mu(x) & 2\mathrm{i}|\Delta|\left(-\mathrm{i}\frac{d}{dx} + \mathrm{i}p_y\right) \\ -2\mathrm{i}|\Delta|\left(-\mathrm{i}\frac{d}{dx} - \mathrm{i}p_y\right) & \mu(x) \end{pmatrix}, \tag{6.1.6}$$

where we have ignored the quadratic terms in p, and p_y is a constant parameter, not an operator. This is a quasi-1D Hamiltonian that can be solved for each value of p_y independently. We propose an ansatz for the gapless interface states:

$$|\psi_{p_y}(x, y)\rangle = e^{\mathrm{i}p_y y} \exp\left(-\frac{1}{2|\Delta|}\int_0^x \mu(x')dx'\right)|\phi_0\rangle \tag{6.1.7}$$

for a constant, normalized spinor $|\phi_0\rangle$. The secular equation for a zero-energy mode at $p_y = 0$ is

$$\mathcal{H}_{\text{BdG}}(0)|\psi_0(x, y)\rangle = 0 \qquad \Longrightarrow \begin{pmatrix} -\mu(x) & -\mu(x) \\ \mu(x) & \mu(x) \end{pmatrix}|\phi_0\rangle = 0. \tag{6.1.8}$$

The constant spinor which is a solution of this equation is $|\phi_0\rangle = 1/\sqrt{2}\,(1, -1)^{\mathsf{T}}$. This form of the constant spinor immediately simplifies the solution of the problem at finite p_y. We see that the term proportional to p_y in Eq. (6.1.6) is $-2|\Delta|p_y\sigma_x$. Since $\sigma_x|\phi_0\rangle = -|\phi_0\rangle$, i.e., the solution $|\phi_0\rangle$ is an eigenstate of σ_x, we conclude that $|\psi_{p_y}(x, y)\rangle$ is an eigenstate of $\mathcal{H}_{\text{BdG}}(p_y)$ with energy $E(p_y) = -2|\Delta|p_y$. Thus, we have found a normalizable bound state solution at the interface of two regions with $\mu < 0$ and $\mu > 0$ respectively. This set of bound states, parameterized by the conserved quantum number p_y is gapless and chiral, i.e., the group velocity of the quasiparticle dispersion is always negative and never changes sign (in this simplified model). The chirality is determined by the sign of the "spectral" Chern number mentioned above which we will calculate below.

These gapless edge states have quite remarkable properties and are not the same chiral complex fermions that propagate on the edge of integer quantum Hall states, but chiral real (Majorana) fermions. Using Clifford algebra representation theory it can be shown that the so-called chiral Majorana (or Majorana-Weyl) fermions can only be found in spacetime dimensions $(8k + 2)$, where $k = 0, 1, 2, \cdots$. Thus, we can only find chiral-Majorana states in $(1+1)$ dimensions or in $(9+1)$ dimensions (or higher!). In condensed matter, we are stuck with $(1+1)$ dimensions where we have now seen that they appear as the boundary states of chiral topological superconductors. The simplest interpretation of such chiral Majorana fermions is as half of a conventional chiral fermion, i.e., its real or imaginary part. To show this, we will consider the edge state of a Chern number 1 quantum Hall system for a single edge

$$\mathcal{H}^{(\text{QH})}_{\text{edge}} = \hbar v \sum_p p\, \eta_p^\dagger \eta_p, \tag{6.1.9}$$

where p is the momentum along the edge. The fermion operators satisfy $\left\{\eta_p^\dagger, \eta_{p'}\right\} = \delta_{pp'}$. Similar to the discussion on the 1D superconducting wire we can decompose these operators into their real and imaginary Majorana parts

$$\eta_p = \frac{1}{2}(\gamma_{1,p} + \mathrm{i}\gamma_{2,p}), \qquad \eta_p^\dagger = \frac{1}{2}(\gamma_{1,-p} - \mathrm{i}\gamma_{2,-p}), \tag{6.1.10}$$

where $\gamma_{a,p}$ $(a = 1, 2)$ are Majorana fermion operators satisfying $\gamma_{a,p}^\dagger = \gamma_{a,-p}$ and $\left\{\gamma_{a,-p}, \gamma_{b,p'}\right\} = 2\delta_{ab}\delta_{pp'}$. The quantum Hall edge Hamiltonian now becomes

$$\begin{aligned}
\mathcal{H}_{\text{edge}}^{(\text{QH})} =& \hbar v \sum_{p\geq 0} p(\eta_p^\dagger \eta_p - \eta_{-p}^\dagger \eta_{-p}) \\
=& \frac{\hbar v}{4} \sum_{p\geq 0} p \left\{(\gamma_{1,-p} - \mathrm{i}\gamma_{2,-p})(\gamma_{1,p} + \mathrm{i}\gamma_{2,p}) - (\gamma_{1,p} - \mathrm{i}\gamma_{2,p})(\gamma_{1,-p} + \mathrm{i}\gamma_{2,-p})\right\} \\
=& \frac{\hbar v}{4} \sum_{p\geq 0} p \left(\gamma_{1,-p}\gamma_{1,p} + \gamma_{2,-p}\gamma_{2,p} - \gamma_{1,p}\gamma_{1,-p} - \gamma_{2,p}\gamma_{2,-p}\right) \\
=& \frac{\hbar v}{2} \sum_{p\geq 0} p \left(\gamma_{1,-p}\gamma_{1,p} + \gamma_{2,-p}\gamma_{2,p} - 2\right).
\end{aligned} \tag{6.1.11}$$

Thus

$$\mathcal{H}_{\text{edge}}^{(\text{QH})} = \frac{\hbar v}{2} \sum_{p\geq 0} p \left(\gamma_{1,-p}\gamma_{1,p} + \gamma_{2,-p}\gamma_{2,p}\right) \tag{6.1.12}$$

up to a constant shift of the energy. This Hamiltonian is exactly two copies of a chiral Majorana Hamiltonian. The edge/domain-wall fermion Hamiltonian of the chiral p-wave superconductor will be

$$\mathcal{H}_{\text{edge}}^{(p-\text{wave})} = \frac{\hbar v}{2} \sum_{p\geq 0} p\gamma_{-p}\gamma_p. \tag{6.1.13}$$

Finding gapless states on a domain wall of μ is an indicator that the phases with $\mu > 0$ and $\mu < 0$ are distinct. If they were the same phase of matter we should be able to adiabatically connect these states continuously. However, we have shown a specific case of the more general result that any interface between a region with $\mu > 0$ and a region with $\mu < 0$ will have gapless states that generate a discontinuity in the interpolation between the two regions. The question remaining is: Is $\mu > 0$ or $\mu < 0$ non-trivial? The answer is that we have a trivial superconductor for $\mu < 0$ (adiabatically continued to $\mu \to -\infty$) and a topological superconductor for $\mu > 0$. Remember that for now we are only considering μ in the neighborhood of 0 and using the continuum model expanded around $(p_x, p_y) = (0, 0)$. We will now define a bulk topological invariant for 2D superconductors that can distinguish the trivial superconductor state from the chiral topological superconductor state. For the spinless Bogoliubov-deGennes Hamiltonian, which is of the form

$$H_{\text{BdG}} = \frac{1}{2} \sum_{\boldsymbol{p}} \Psi_{\boldsymbol{p}}^\dagger \left[\boldsymbol{d}(\boldsymbol{p}, \mu) \cdot \boldsymbol{\sigma}\right] \Psi_{\boldsymbol{p}}, \tag{6.1.14a}$$

$$\boldsymbol{d}(\boldsymbol{p}, \mu) = \left(-2|\Delta| p_y, -2|\Delta| p_x, p^2/2m - \mu\right), \tag{6.1.14b}$$

the topological invariant is the spectral Chern number which simplifies, for this Hamiltonian, to the winding number

$$\mathsf{C}^{(1)} = \frac{1}{8\pi} \int \mathrm{d}^2\boldsymbol{p}\, \epsilon^{ij}\, \hat{\boldsymbol{d}} \cdot \left(\partial_{p_i}\hat{\boldsymbol{d}} \times \partial_{p_j}\hat{\boldsymbol{d}}\right) = \frac{1}{8\pi} \int \mathrm{d}^2\boldsymbol{p}\, \frac{\epsilon^{ij}}{|\boldsymbol{d}|^3}\, \boldsymbol{d} \cdot \left(\partial_{p_i}\boldsymbol{d} \times \partial_{p_j}\boldsymbol{d}\right). \tag{6.1.15}$$

We defined the unit vector $\hat{\boldsymbol{d}} = \boldsymbol{d}/|\boldsymbol{d}|$, which is possible since $|\boldsymbol{d}| \neq 0$ due to the existence of a gap. This integral has a special form and is equal to the degree of the mapping from momentum space onto the 2-sphere S^2 given by $\hat{d}_1^2 + \hat{d}_2^2 + \hat{d}_3^2 = 1$. As it stands, the degree of the mapping $\hat{\boldsymbol{d}} : \mathbb{R}^2 \to S^2$ is not well-defined because the domain is not compact, i.e., (p_x, p_y) is only restricted to lie in the Euclidean plane ($\mathbb{R}^2$). However, for our choice of the map $\hat{\boldsymbol{d}}$ we can define the winding number by choosing an equivalent, but compact, domain. To understand the necessary choice of domain we can simply look at the explicit form of $\hat{\boldsymbol{d}}(\boldsymbol{p})$

$$\hat{\boldsymbol{d}}(\boldsymbol{p}) = \frac{(-2|\Delta|p_y, -2|\Delta|p_x, p^2/2m - \mu)}{\sqrt{4|\Delta|^2 p^2 + (p^2/2m - \mu)^2}}. \tag{6.1.16}$$

We see that $\lim_{|\boldsymbol{p}|\to\infty} \hat{\boldsymbol{d}}(\boldsymbol{p}) = (0,0,1)$ and it does not depend on the direction in which we take the limit in the 2D plane. Because of the uniqueness of this limit we are free to perform the *one-point compactification* of $\mathbb{R}^2$ which amounts to including the point at infinity in our domain. The topology of $\mathbb{R}^2 \cup \{\infty\}$ is the same as S^2 and thus we can consider the degree of our map from the compactified momentum space (S^2) to the unit $\hat{\boldsymbol{d}}$-vector space (S^2). Using the explicit form of the $\hat{\boldsymbol{d}}$-vector for this model, we find

$$\mathsf{C}^{(1)} = \frac{1}{\pi} \int \mathrm{d}^2\boldsymbol{p} \frac{|\Delta|^2 \left(\frac{p^2}{2m} + \mu\right)}{\left[4|\Delta|^2 p^2 + \left(\frac{p^2}{2m} - \mu\right)^2\right]^{3/2}}. \tag{6.1.17}$$

The evaluation of this integral can be easily carried out numerically. The result is $\mathsf{C}^{(1)} = 0$ for $\mu < 0$ and $\mathsf{C}^{(1)} = 1$ for $\mu > 0$, i.e., there are two different phases separated by a quantum critical point at $\mu = 0$. Thus we have identified the phase which is in the chiral superconductor state to be $\mu > 0$.

6.2 Argument for the existence of Majorana bound states on vortices

A simple but rigorous argument can show us the presence of zero energy bound states in the core of vortices in a superconductor. Assume we have a chiral ($p + ip$) superconductor in two geometries: a disk with an edge and a cylinder with two edges. Since it is a topological superconductor, the system will have chiral dispersing (Majorana) gapless modes along the edges. In Fig. 6.1, the spectra are plotted versus the momentum along the edge, and they are qualitatively very different in the two cases. For an edge of length L, the smallest difference between two momenta along the edge is $2\pi/L$. The energy difference between two levels is $v2\pi/L$, where v is the velocity of the edge mode.

In a single particle superconducting Hamiltonian the number of total single-particle eigenvalues is always even. This is clear from the fact that whatever the spinor of the nonsuperconducting Hamiltonian is, when superconductivity is added, we have a doubled spectrum, so that every energy state at $E > 0$ comes with a counterpart at energy $-E$. When labeled by momentum quantum number, for a system with just one edge, like the disk, there cannot be a single state at momentum $p = 0$ at energy $E = 0$. If such a state was there, the spectrum would contain an odd number of states. Hence the spectrum of the linearized edge mode cannot have a state at $E = 0$, $p = 0$ on the disk. The one way to introduce such a state is to have antiperiodic boundary conditions, with the spectrum of the edge being at momenta $\pi(2n+1)/L$, $n \in \mathbb{Z}$. On the cylinder, as two edges are present, periodic boundary conditions are allowed (as are antiperiodic, which can be obtained by threading a flux through the cylinder).

We now add a single vortex inside the disk, far away from the edge of the disk. What is the influence of the vortex on the edge? The vortex induces a phase 2π in the units of the superconducting quantum $hc/2e$, which means that the phase of Δ changes by 2π, and that

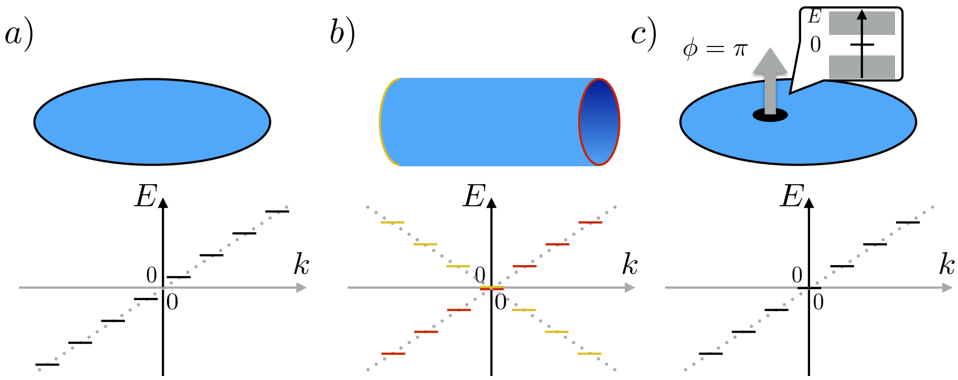

Figure 6.1: Spectra of a Chiral superconductor in different geometries: (a) disk, (b) cylinder, and (c) disk with flux defect. Shown are the spectra of the chiral topological boundary modes including their finite-size quantization with level spacing $v2\pi/L$. If a π flux is inserted in the disk geometry (c), it binds an isolated zero-energy state. At the same time, a single zero-energy state appears on the edge.

of the electronic operators by π upon a full rotation around the edge. This implies that the antiperiodic boundary conditions on the edge without vortex changes to periodic boundary conditions in the presence of the vortex. The spectrum on the edge then is translated by π/L compared to the case without the vortex, making it have an energy level at $p = 0, E = 0$. This would mean that the spectrum has an odd number of levels. However, this cannot be true, as we explained above, since the number of levels is always even. We are hence missing one unpaired level. Where is it? Since the only difference from the case with no vortex is the vortex itself, we draw the conclusion that the missing level is associated with the vortex, and is a bound state on the vortex. We also draw the conclusion that, since it is unpaired and really bound to the vortex, it has to rest exactly at $E = 0$, thereby showing that chiral superconductors have Majorana zero modes in their vortex core.

6.3 Vortices in two-dimensional chiral p-wave superconductors

6.3.1 Explicit bound state solutions

Let us explicitly show that a vortex in a chiral superconductor will contain a zero mode. This calculation, which is a variant of our calculation for the existence of a Majorana mode at the interface between a topological and a trivial superconductor. We consider a disk of radius R which has $\mu > 0$ surrounded by a region with $\mu < 0$ for $r > R$. We know from our previous discussion that there will be a single branch of chiral Majorana states localized near $r = R$, but no exact zero mode. If we take the limit $R \to 0$ this represents a vortex and all the low-energy modes on the interface will be pushed to higher energies. If we put a π-flux inside the trivial region it will change the boundary conditions such that even in the $R \to 0$ limit there will be a zero-mode in the spectrum localized on the vortex.

Now let us take the Bogoliubov-deGennes Hamiltonian in the Dirac limit ($m \to \infty$) and solve the Bogoliubov-deGennes equations in the presence of a vortex located at $r = 0$ in the disk geometry in polar coordinates. Let $\Delta(r, \vartheta) = |\Delta(r)|e^{\mathrm{i}\alpha(\boldsymbol{r})}$. The profile $|\Delta(r)|$ for a vortex will depend on the details of the model, but must vanish inside the vortex core region, e.g., for an infinitely thin core we just need $|\Delta(0)| = 0$. We take the phase $\alpha(\boldsymbol{r})$ to be equal to the polar angle ϑ at $\boldsymbol{r}$.

The first step in the solution of the bound state for this vortex profile is to gauge transform the

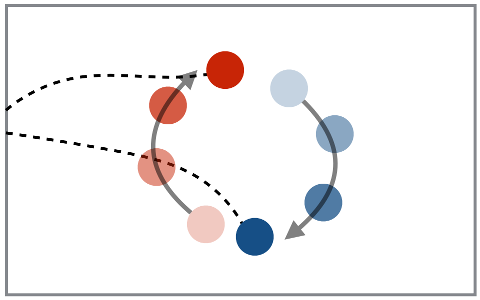

Figure 6.2: Illustration of the exchange of two vortices in a chiral p-wave superconductor. The dotted lines represent branch cuts across which the phase of the superconducting order parameter jumps by 2π.

phase of $\Delta(r,\vartheta)$ into the fermion operators via $\Psi(\boldsymbol{r}) \to e^{\mathrm{i}\alpha(\boldsymbol{r})/2}\Psi(\boldsymbol{r})$. This has two effects: (i) it simplifies the solution of the Bogoliubov-deGennes differential equations and (ii) converts the boundary conditions of $\Psi(\boldsymbol{r})$ from periodic to anti-periodic around the vortex position $\boldsymbol{r}=0$. In polar coordinates the remaining single-particle Bogoliubov-deGennes Hamiltonian is simply

$$\mathcal{H}_{\mathrm{BdG}} = \frac{1}{2}\begin{pmatrix} -\mu & 2|\Delta(r)|e^{\mathrm{i}\vartheta}\left(\frac{\partial}{\partial r}+\frac{\mathrm{i}}{r}\frac{\partial}{\partial\vartheta}\right) \\ -2|\Delta(r)|e^{-\mathrm{i}\vartheta}\left(\frac{\partial}{\partial r}-\frac{\mathrm{i}}{r}\frac{\partial}{\partial\vartheta}\right) & \mu \end{pmatrix}. \tag{6.3.1}$$

We want to solve $\mathcal{H}_{\mathrm{BdG}}\Psi = E\Psi = 0$ which we can do with the ansatz

$$\Psi_0(r,\vartheta) = \frac{\mathrm{i}}{\sqrt{r\mathcal{N}}}\exp\left[-\frac{1}{2}\int_0^r \frac{\mu(r')}{|\Delta(r')|}\mathrm{d}r'\right]\begin{pmatrix} -e^{\mathrm{i}\vartheta/2} \\ e^{-\mathrm{i}\vartheta/2}\end{pmatrix} \equiv \mathrm{i}g(r)\begin{pmatrix} -e^{\mathrm{i}\vartheta/2} \\ e^{-\mathrm{i}\vartheta/2}\end{pmatrix}, \tag{6.3.2}$$

where $\mathcal{N}$ is a normalization constant. The function $g(r)$ is localized at the location of the vortex. We see that $\Psi_0(r,\vartheta+2\pi) = -\Psi_0(r,\vartheta)$ as required. From an explicit check one can see that $\mathcal{H}_{\mathrm{BdG}}\Psi_0(r,\vartheta)=0$. The field operator which annihilates fermion quanta in this localized state is

$$\gamma = \int r\mathrm{d}r\mathrm{d}\vartheta\ \mathrm{i}g(r)\left[-e^{\mathrm{i}\vartheta/2}c(r,\vartheta) + e^{-\mathrm{i}\vartheta/2}c^\dagger(r,\vartheta)\right], \tag{6.3.3}$$

from which we can immediately see that $\gamma=\gamma^\dagger$. Thus the vortex traps a single Majorana bound state at zero-energy.

6.3.2 Non-Abelian statistics of vortices in chiral p-wave superconductors

We have shown in the last Section that on each vortex in a spinless chiral superconductor there exists a single Majorana bound state. If we have a collection of $2N$ vortices which are well-separated from each other, a low-energy subspace is generated which in the thermodynamic limit leads to a ground state degeneracy of 2^N. For example, two vortices give a degeneracy of 2, which can be understood by combining the two localized Majorana bound states into a single complex fermion state which can be occupied or un-occupied, akin to the end states of the superconducting wire. From $2N$ vortices one can form N complex fermion states giving a degeneracy of 2^N, which can be broken up into the subspace of 2^{N-1} states with even fermion parity and the 2^{N-1} states with odd fermion parity. As an aside, since we have operators that mutually anti-commute and square to $+1$ we can define a Clifford algebra operator structure using the set of $2N$ γ_i.

Let us begin with a single pair of vortices which have localized Majorana operators γ_1 and γ_2 respectively and are assumed to be well separated. We imagine that we adiabatically move the vortices in order to exchange the two Majorana fermions. If we move them slow enough then the only outcome of exchanging the vortices is a unitary operator acting on the two degenerate states which make up the ground state subspace. If we exchange the two vortices then we have $\gamma_1 \to \gamma_2$ and $\gamma_2 \to \gamma_1$. However if we look at Fig. 6.2 we immediately see there is a complication. In this figure we have illustrated the exchange of two vortices and the dotted lines represent branch cuts across which the phase of the superconductor order parameter jumps by 2π. Since our solution of the Majorana bound states used the gauge transformed fermion operators we see that the bound state on the red vortex, which passes through the branch cut of the blue vortex, picks up an additional minus sign upon exchange. Thus the exchange of two vortices is effected by

$$\gamma_1 \to \gamma_2, \qquad \gamma_2 \to -\gamma_1. \tag{6.3.4}$$

In general, if we have $2N$ vortices, we can think of the different exchange operators $T_{ij}(\gamma_a)$ which for our choice of conventions send $\gamma_i \to \gamma_j$, $\gamma_j \to -\gamma_i$, and $\gamma_k \to \gamma_k$ for all $k \neq i, j$. A concept for topological quantum computation arises from using the degenerate Hilbert space spanned by the Majorana particles as quantum bits (qubits) and the exchange operations as multi-qubit gates. Importantly, since the exchange operations preserve Fermion parity (we assume they are adiabatic, so there is no level crossing) we can only work in either the even parity or odd parity subspace and use it as qubits. Let us consider the case of 4 Majoranas, which span a ($\sqrt{2}^4 = 4$)-dimensional Hilbert space. Let us focus on the even-parity sector, which is two-dimensional and consists of the states $|00\rangle$ and $|11\rangle$ where the two occupation numbers are those of the fermions with creation and annihilation operators defined as follows

$$\begin{aligned} a &= \frac{1}{2}(\gamma_1 + \mathrm{i}\gamma_2), \qquad a^\dagger = \frac{1}{2}(\gamma_1 - \mathrm{i}\gamma_2), \\ b &= \frac{1}{2}(\gamma_3 + \mathrm{i}\gamma_4), \qquad b^\dagger = \frac{1}{2}(\gamma_3 - \mathrm{i}\gamma_4). \end{aligned} \tag{6.3.5}$$

Consider now a *double-exchange* of Majoranas γ_2 and γ_4, which results in $\gamma_2 \to -\gamma_2$ and $\gamma_4 \to -\gamma_4$. As a result, $a \leftrightarrow a^\dagger$ and $b \leftrightarrow b^\dagger$. Thus, an initially empty state $|00\rangle$ would now be measured as occupied under both operators a and b and thus the double-exchange of the Majoranas results in $|00\rangle \leftrightarrow |11\rangle$, i.e., a flip of our qubit, also called a Pauli-x gate (because of the matrix representation of the operation in the $|00\rangle, |11\rangle$ basis being the first Pauli matrix). Importantly, the operation defined in Eq. (6.3.4) does not commute with this double-exchange of γ_2 and γ_4, an indication of the *non-Abelian* statistics of Majorana fermions. This property allows for topologically protected, albeit not universal, quantum computation operations.

6.4 The 16-fold way classification of two-dimensional chiral superconductors

We have now noticed that there are two characterizations of a topological superconductor, but they are seemingly different. First, the spectral Chern number is an integer $\mathsf{C}^{(1)} \in \mathbb{Z}$. Directly related to it is the number of chiral Majorana modes on the edge, which in turn is related to an experimental observable, the thermal conductivity on the edge. Hence the system has a $\mathbb{Z}$ index, which becomes obvious when an edge exists. We then saw that a $(p+\mathrm{i}p)$ superconductor (i.e., a topological superconductor with Chern number equal to one) with a vortex threaded through it exhibits a Majorana zero energy mode at the core of the vortex. A $(d+\mathrm{i}d)$ superconductor, with Chern number equal to 2, would exhibit two Majorana modes in the core of the vortex. However, those two Majorana modes would be unstable towards single particle hybridization terms, which would push them away from zero energy, and leave the core of the vortex with no states in it. The generalization tells us that an even Chern number topological superconductor has no Majorana zero modes in the vortex while an odd Chern number topological superconductor has

one Majorana zero mode in its core. This shows that the defects (vortices) in a topological superconductor are classified by a $\mathbb{Z}_2$ number ($\mathsf{C}^{(1)} \mod 2$).
We now show that there is a third classification related to the idea of topological order. In the absence of an edge and in the absence of vortex defects, there is a $\mathbb{Z}_{16}$ classification of topological superconductors indexed by $\mathsf{C}^{(1)} \mod 16$, which can be put on solid grounds by the formalism of topological quantum field theory (TQFT).
We ask how we can classify the system in the absence of an edge. One way would be to compute the phases that wavefunctions can acquire upon taking particles or quasiparticles around each other. However, the system is made out of electrons (its a superconductor), so usually nothing special can happen to phases of electrons. The only "special" excitation of the superconductor is a vortex, so we will look at the phase that two vortices acquire upon exchange. We can calculate this with an argument. Take two copies of the $(p + \mathrm{i}p)$ superconductor governed by the Hamiltonian

$$H = \frac{\mathrm{i}}{4} \sum_{j,k} A_{jk} (\gamma_{1,j}\gamma_{1,k} + \gamma_{2,j}\gamma_{2,k}), \tag{6.4.1}$$

written in terms of Majorana operators $\gamma_{1,j}$ for one copy and $\gamma_{2,j}$ for the other copy. These operators can be combined into an complex fermion $c_j = (\gamma_{1,j} + \mathrm{i}\gamma_{2,j})/2$ in terms of which the Hamiltonian becomes

$$H = \mathrm{i} \sum_{j,k} A_{jk} c_j^\dagger c_k. \tag{6.4.2}$$

This Hamiltonian has a "fake" U(1) symmetry given by our choice of A_{jk} for both Hamiltonians. (Since the system is gapped, we expect our universal conclusions to hold even when this symmetry is stripped away). Thus, the system is a quantum Hall state of Hall conductance $\mathsf{C}^{(1)}$ (in units of e^2/h) if each of the superconductors had Chern number $\mathsf{C}^{(1)}$. We now ask what happens when we thread a superconducting vortex $h/2e$, which is equal to π. Threading a flux 2π in a quantum Hall state of Chern number $\mathsf{C}^{(1)}$ pulls $\mathsf{C}^{(1)}$ electron charges to the vortex core through the Hall effect, hence a π flux pulls $\mathsf{C}^{(1)}/2$ electron charges towards the core. We then try to compute the phase acquired when a vortex is exchanged with another vortex. This is an exchange process, which is half a braid. A braid of two vortices is equivalent to $\mathsf{C}^{(1)}/2$ electrons braided with a π vortex, giving rise to a phase $\pi\mathsf{C}^{(1)}/2$ upon a braid, and $\pi\mathsf{C}^{(1)}/4$ under exchange. Since this is the phase for exchange of vortices in two exactly identical superimposed superconductors, the phase for exchange in one of them is half that, $\pi\mathsf{C}^{(1)}/8 = 2\pi\mathsf{C}^{(1)}/16$. This shows that the phase for vortex exchange is defined only mod 16.
Let us summarize what we have learned about the vortices in chiral superconductors with odd Chern number. We have seen that well-separated vortices hold a Majorana zero mode at their core. When these vortices come together, the two Majorana modes hybridize and split, giving rise to two states which differ by their fermion parity. Let us call the Bogoliubov-deGennes vacuum 1 and the Bogoliubov quasiparticle ψ, and the Majorana fermion of the vortex σ. We can then formalize the fusion of two vortices by writing down a fusion rule

$$\sigma \times \sigma = 1 + \psi, \tag{6.4.3}$$

which basically tells us that combining two Majoranas can either go to a state with no fermion or at one with a fermion – the fermion parity (and density) would be different for the two states. Which one it is depends on the microscopics of the model. Hence a quantum state of two Majoranas has to be described by another quantum number, which describes the "fusion channel" of those two Majoranas – either the vacuum or the Bogoliubov quasiparticle. The fusion rule (6.4.3) allows for multiple fusion channels.This is a manifestation of the fact that the Majoranas are non-Abelian anyons. When two Bogoliubov quasiparticles fuse, they condense (form a Cooper pair) and go to the vacuum

$$\psi \times \psi = 1, \tag{6.4.4}$$

while the fusion of a Bogoliubov and a Majorana quasiparticle basically creates another Majorana

$$\psi \times \sigma = \sigma. \tag{6.4.5}$$

This can be rationalized by thinking of the complex Bogoliubov quasiparticle as made out of two Majoranas which then couple to the third Majorana. The Hamiltonian is a 3×3 antisymmetric matrix that necessarily has a zero eigenvalue which is another Majorana fermion coming as a result of the fusion.

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