Week 13
Lecture Notes:
Topological Condensed Matter Physics

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Chapter 13

The fractional quantum Hall effect II

Learning goals

- We know what a coherent state path integral is.
- We know the concept of a composite fermion.
- We know how to get from composite fermions to a Chern-Simons theory.

- Willett, R. et al., Phys. Rev. Lett. **5**9, 1776 (1987)

13.1 Path integrals

13.1.1 Why do we need a path integral

In this section we try to argue why we need a path integral representation of the partition sum

$$Z = \int D[\bar{\phi}\phi] e^{-S[\bar{\phi},\phi]}. \tag{13.1.1}$$

First of all, we trade non-commuting bosonic *operators* with an integral over all "field" configurations, i.e.,

$$[.,.] \to D[\bar{\phi},\phi]. \tag{13.1.2}$$

Moreover, we replace complicated anti-commutations for fermions we a simple tool called Grassmann numbers. Before we are going to explain what we exactly mean with expression (13.1.1), we list a few nice properties that we will gain from a path integral formalism.

1. We can use Gaussian integrals

$$\int D[\bar{\phi}\phi] e^{-\bar{\phi}^{\mathrm{T}} A \phi} = \frac{1}{\det A}. \tag{13.1.3}$$

2. We can complete the square

$$\int D[\bar{\phi}\phi] e^{-\bar{\phi}^{\mathrm{T}} A\phi + \vartheta^{\mathrm{T}}\bar{\phi} + \bar{\vartheta}^{\mathrm{T}}\phi} = \int D[\bar{\phi}\phi] e^{-(\bar{\phi} - A^{-1}\bar{\vartheta})^{\mathrm{T}} A(\phi - A^{-1}\vartheta) + \bar{\vartheta}^{\mathrm{T}} A^{-1}\vartheta} = \frac{e^{\bar{\vartheta}^{\mathrm{T}} A^{-1}\vartheta}}{\det A}. \tag{13.1.4}$$

This completing of the square in turn has three important applications:

(a) Greens functions (or more generally, two-point correlators) in a quadratic theory can be calculated by coupling sources ϑ

$$\langle \bar{\phi}_i \phi_j \rangle = \frac{\int D[\bar{\phi}\phi] \bar{\phi}_i \phi_j e^{-S[\bar{\phi},\phi]}}{\int D[\bar{\phi}\phi] e^{-S[\bar{\phi},\phi]}} = \frac{\delta^2}{\delta\vartheta_i \delta\bar{\vartheta}_j}\Big|_{\vartheta=\bar{\vartheta}=0} e^{\bar{\vartheta}^{\mathrm{T}} A^{-1}\vartheta} = [A^{-1}]_{ij}. \tag{13.1.5}$$

(b) "Integrating out" linearly coupled quadratic degrees of freedom

$$\int D[\bar{\phi}\phi]D[\bar{\vartheta}\vartheta]e^{-S[\bar{\phi},\phi]+\phi^{\mathrm{T}}\bar{\vartheta}+\bar{\phi}^{\mathrm{T}}\vartheta-\bar{\vartheta}^{\mathrm{T}}B\vartheta} = \int D[\bar{\phi}\phi]e^{-S[\bar{\phi},\phi]+\bar{\phi}^{\mathrm{T}}B^{-1}\phi} = \int D[\bar{\phi}\phi]e^{-S_{\mathrm{eff}}[\bar{\phi},\phi]}. \quad (13.1.6)$$

(c) Or the reverse of it, called *Hubbard Stratonovich transformation*

$$\int D[\bar{\phi}\phi]e^{-\bar{\phi}^{\mathrm{T}}A\phi+\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi} = \int D[\bar{\phi}\phi]D[\vartheta]e^{-(\bar{\vartheta}-\bar{\phi}^{\mathrm{T}}\phi)(\vartheta-\bar{\phi}^{\mathrm{T}}\phi)-\phi^{\mathrm{T}}A\phi+\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi} \quad (13.1.7)$$

$$= \int D[\bar{\phi}\phi]D[\vartheta]e^{-\bar{\vartheta}\vartheta+2\vartheta\bar{\phi}^{\mathrm{T}}\phi-\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi-\bar{\phi}^{\mathrm{T}}A\phi+\bar{\phi}^{\mathrm{T}}\phi\bar{\phi}^{\mathrm{T}}\phi} \quad (13.1.8)$$

$$= \int D[\bar{\phi}\phi]D[\vartheta]e^{-\bar{\vartheta}\vartheta-\bar{\phi}^{\mathrm{T}}(A+2\vartheta)\phi} \quad (13.1.9)$$

$$= \int D[\vartheta]e^{-\bar{\vartheta}\vartheta-\mathrm{tr}\log[A+2\vartheta]}. \quad (13.1.10)$$

This is still not a quadratic theory, but the logarithm can be expanded step by step to get an effective theory.

3. We can do mean-field calculations

$$\frac{\delta S[\bar{\phi},\phi]}{\delta\bar{\phi}} = 0 \qquad \Rightarrow \qquad \phi_{\mathrm{MF}}. \quad (13.1.11)$$

After all these expected profits, let us start introducing such a path integral representation of the partition sum.

13.1.2 Coherent state path integral

Given a quantum mechanical problem defined by a Hamiltonian H, we want to express the partition sum

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_n \langle m|e^{-\beta H}|m\rangle, \quad (13.1.12)$$

as a path integral. For this we use coherent states

$$|\phi\rangle = e^{\eta\sum_i \phi_i c_i^\dagger}|\mathrm{vac}\rangle \quad \Rightarrow c_i|\phi\rangle = \phi_i|\phi\rangle, \quad (13.1.13)$$

and we used $\eta = \pm 1$ for bosons (fermions), respectively. Remember that they are not orthogonal

$$\langle\phi|\vartheta\rangle = e^{\bar{\phi}^{\mathrm{T}}\vartheta}. \quad (13.1.14)$$

For bosons, $\phi_i \in \mathbb{C}$. For fermions we need to take care of anti-commutations. This can be achieved by requiring ϕ_i to be Grassmann numbers.
Grassmann numbers are defined by

$$\phi_i\phi_j = -\phi_j\phi_i; \quad \partial_{\phi_i}\phi_j = 0; \quad \int d\phi_i = 0; \quad \int d\phi_i\phi_i = 1. \quad (13.1.15)$$

From this follows immediately

$$\int d\bar{\phi}_i d\phi_i\, e^{-\bar{\phi}_i a\phi_i} = \int d\bar{\phi}_i d\phi_i\, [1-\phi_i\phi_i a] = a. \quad (13.1.16)$$

Which immediately leads to

$$\int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}} A\phi} = \prod_n \int d\bar{\phi}_n d\phi_n\, e^{-\sum_{rs} \bar{\phi}_r A_{rs} \phi_s} = \det A. \tag{13.1.17}$$

Note that this is similar to the bosonic case, however $[\det A]^{-1}$ is replaced with $\det A$.
With the help of the coherent states $|\phi\rangle$ we can now write a complicated but tremendously useful resolution of the unity

$$\mathbb{1} = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi|. \tag{13.1.18}$$

To prove this identity, we have to show that c_i and $c_i^\dagger$ commute with the right-hand side:

$$c_i \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi| = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} c_i|\phi\rangle\langle\phi| = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \phi_i|\phi\rangle\langle\phi| \tag{13.1.19}$$

$$= -\int d(\bar{\phi}\phi)\, [\partial_{\bar{\phi}_i} e^{-\bar{\phi}^{\mathrm{T}}\phi}] |\phi\rangle\langle\phi| \tag{13.1.20}$$

$$\overset{\text{P.I.}}{=} \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} [\underbrace{(\partial_{\bar{\phi}_i}|\phi\rangle)}_{=0}\langle\phi| + |\phi\rangle(\partial_{\bar{\phi}_i}\langle\phi|)] \tag{13.1.21}$$

$$= \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi| c_i. \tag{13.1.22}$$

In the last line we used

$$c_i^\dagger|\phi\rangle = \partial_{\phi_i}|\phi\rangle \qquad \Rightarrow \qquad \partial_{\phi_i}\langle\phi| = \langle\phi_i|a_i. \tag{13.1.23}$$

With this we showed that c_i indeed commutes with the alleged unity. For $c_i^\dagger$ one starts from the other end end and goes through the same manipulations (show!). As all operators in the Fock space can be written as products (and sums) of the creation and annihilation operators, we have shown that indeed

$$\int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} |\phi\rangle\langle\phi| \propto \mathbb{1}. \tag{13.1.24}$$

Let us check for the proportionality factor

$$\langle \mathrm{vac}|\mathbb{1}|\mathrm{vac}\rangle = 1 = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \langle \mathrm{vac}|\phi\rangle\langle\phi|\mathrm{vac}\rangle. \tag{13.1.25}$$

Let us now rewrite the trace in the partition sum

$$Z = \sum_n \langle n|e^{-\beta H}|n\rangle = \int d(\bar{\phi}\phi) \sum_n \langle n|\phi\rangle\langle\phi|e^{-\beta H}|n\rangle e^{-\bar{\phi}^{\mathrm{T}}\phi} \tag{13.1.26}$$

$$= \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \sum_n \langle \eta\phi|n\rangle\langle n|e^{-\beta H}|\phi\rangle = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \langle\eta\phi|e^{-\beta H}|\phi\rangle. \tag{13.1.27}$$

Now we need to fix an important property. In order for our path integral approach to go through, we need to normal order our Hamiltonian. This means, we arrange all operators in H such that all $c_i^\dagger$ stand to the left of all c_i. As the fields ϕ_i are just complex numbers (for bosons, at least), this will be the last time we can take care of the operator nature of second quantized quantum mechanics. We write for the normal ordered Hamiltonian explicitly

$$Z = \int d(\bar{\phi}\phi)\, e^{-\bar{\phi}^{\mathrm{T}}\phi} \langle\eta\phi|e^{-\beta H(c^\dagger, c)}|\phi\rangle. \tag{13.1.28}$$

Next, we re-write

$$\beta H(c^\dagger, c) = \frac{\beta}{N}\sum_{i=1}^{N} H(c^\dagger, c) \tag{13.1.29}$$

and we insert a unity in between all resulting factors

$$Z = \int_{\phi^1=\eta\phi^N,\,\bar{\phi}^1=\eta\bar{\phi}^N} \prod_{i=1}^{N} d(\bar{\phi}^i\phi^i) e^{\frac{\beta}{N}\sum_{i=1}^{N}\frac{(\bar{\phi}^i-\bar{\phi}^{i+1})\phi^i}{\beta/N}+H(\bar{\phi}^i,\phi^i)}. \tag{13.1.30}$$

Note that the superscript i labels the i'th insertion of the unity. One often calls β the "imaginary time" in relation to the real time propagator $\exp(itH)$. Within this interpretation, i corresponds to the i'th time slice. If we now take the limit $N \to \infty$, we are taking a continuum limit in imaginary time where

$$\phi^i \to \phi(\tau) \qquad \text{and} \qquad \frac{\beta}{N}\sum_i \to \int_0^\beta d\tau. \tag{13.1.31}$$

We can now write down our sought path integral

$$Z = \int D[\bar{\phi}\phi] e^{-S[\bar{\phi},\phi]}, \tag{13.1.32}$$

$$S[\bar{\phi},\phi] = \int_0^\beta d\tau\, \bar{\phi}^{\mathrm{T}}\partial_\tau\phi + H(\bar{\phi},\phi), \tag{13.1.33}$$

$$D[\bar{\phi},\phi] = \lim_{N\to\infty}\prod_{i=1}^{N} d(\bar{\phi}^i\phi^i); \quad \bar{\phi}(0) = \eta\bar{\phi}(\beta), \;\; \phi(0) = \eta\phi(\beta). \tag{13.1.34}$$

13.1.3 Kubo formula

We already got acquainted with the Kubo formula in Chap. 2. We want to revisit it here in the language of our newly introduced coherent state path integral. Imagine a "force" $F(\boldsymbol{r},\omega)$ coupled to the "coordinate"

$$\hat{X} = \sum_{\alpha\beta} c_\alpha^\dagger X_{\alpha\beta} c_\beta. \tag{13.1.35}$$

We then ask for the linear response coefficient

$$X(\boldsymbol{r},\omega) = \int dr' \chi(\boldsymbol{r}-\boldsymbol{r}',\omega)F(\boldsymbol{r}',\omega). \tag{13.1.36}$$

In path integral formalism the expectation value on the right hand side is expressed as

$$X(\tau) = \sum_{\alpha\beta}\langle\bar{\phi}_\alpha(\tau)X_{\alpha\beta}\phi_\beta(\tau)\rangle_F, \tag{13.1.37}$$

where the subscript F indicates that we have to evaluate this expression in the presence of the force F

$$\delta S_F = \int_0^\beta d\tau\, F(\tau)\bar{\phi}_\alpha(\tau)X_{\alpha\beta}\phi_\beta(\tau). \tag{13.1.38}$$

To generate the expectation value (13.1.37) we can add another ficticious force F' to the action

$$\delta S_{F'} = \int_0^\beta d\tau\, F'(\tau)\bar{\phi}_\alpha(\tau)X'_{\alpha\beta}\phi_\beta(\tau). \tag{13.1.39}$$

With this addition, one can write

$$X(\tau) = -\frac{\delta}{\delta F'(\tau)}\Big|_{F'=0}\log(Z[F,F']). \tag{13.1.40}$$

For the sake of linear response, we imagine F to be small. Therefore, we can apply a Taylor expansion

$$X(\tau) = \int d\tau' \left[\frac{\delta^2}{\delta F'(\tau)\delta F(\tau')}\Big|_{F=F'=0}\log(Z[F,F'])\right]F(\tau') \tag{13.1.41}$$

With this expression we can immediately indentify the linear response coeffiecient. If we assume at $X(\tau)=0$ in the absence of the force

$$\chi(\tau,\tau')=-\frac{1}{Z}\frac{\delta^2}{\delta F'(\tau)\delta F(\tau')}\Big|_{F=F'=0} Z[F,F']. \tag{13.1.42}$$

Electromagnetic response

We consider a system subject to an electromagnetic field $A^\mu=(\mathrm{i}\varphi,\boldsymbol{A})$. The system might react via a redistribution of charge ρ or via an onset of a current $\boldsymbol{j}$. We write $j_\mu=(\mathrm{i}\rho,\boldsymbol{j})$ and look for

$$j_\mu(x)=\int_{t'<t} dx' K_{\mu\nu}(x-x')A^\nu(x'), \tag{13.1.43}$$

where x describes the four-coordinate $(\mathrm{i}t,\boldsymbol{x})$. We remember that we coupled the A^μ-field as $j_\mu A^\mu$ to the Hamiltonian. Therefore,

$$j_\mu=\frac{\delta S}{\delta A^\mu} \qquad \Rightarrow \qquad F=F'=A^\mu. \tag{13.1.44}$$

With this we find

$$K_{\mu\nu}(x-x')=-\frac{1}{Z}\frac{\delta^2}{\delta A^\mu(x)\delta A^\nu(x')}Z[A^\mu]. \tag{13.1.45}$$

Effective theories

If we have a system of charged particles, $H(c^\dagger,c)$, and we are interested in its electro-magnetic response, all we need to know is $K_{\mu\nu}$. In a path integral language, we say we *integrate out the fermions* to obtain an *effective action* in terms of the A^μ-field alone. The peculiar structure of $K_{\mu\nu}$ will fully describe our system in terms of its electro-magnetic system

$$S_{\text{eff}}[A^\mu]=\int_0^\beta d\tau\int dxdx'\, A^\mu(x)K_{\mu\nu}(x-x')A^\nu(x'). \tag{13.1.46}$$

13.2 Composite fermions

13.2.1 From a wave functions to a field theory

In the last chapter we got to know the Laughlin wave function for filling fractions $\nu=\frac{1}{2p+1}$ with $p\in\mathbb{N}$

$$\psi(\{z_i\})=\prod_{i<j}(z_i-z_j)^{\frac{1}{\nu}}e^{-\frac{1}{4}\sum_i|z_i|^2}. \tag{13.2.1}$$

These wave functions are manifestly in the lowest Landau level and in addition to the $(z_i-z_j)^1$ term needed for the Pauli principle there are two (for $\nu=1/3$) more zeros attached to the coincidence of two particles. This observation is identical to attaching $1/\nu-1$ fluxes of 2π to each particle[1]

In the last chapter, we only considered the Laughlin wave function and analyzed its properties. Here, we follow a more ambitious goal. Building on the insight gained through the Laughlin

[1] Up to the fact that a pure flux attachment would require a factor

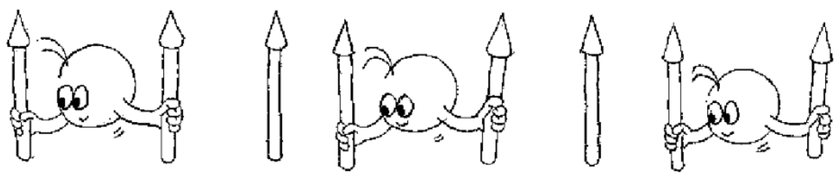
$$e^{-2\mathrm{i}\sum_{i<j}\arg(z_i-z_j)}=\prod_{i<j}\frac{(z_i-z_j)^2}{|z_i-z_j|^2}. \tag{13.2.2}$$

The absence of the factor $1/|z_i-z_j|^2$ in the Laughlin wave function can be seen as the effect of the projection to the lowest Landau level.

wave function, we want to construct an effective theory for the fractional quantum Hall effect including the Hamiltonian! However, we want to assume that the important players are not electrons, but the "bound states of electrons with statistical fluxes" that were at the heart of the Laughlin wave function. In other words, we want to go from an electron wave function (theory), to one of *composite fermions* by

$$\psi(\{\boldsymbol{x}_i\}) \mapsto \psi(\{\boldsymbol{x}_i\}) e^{2\mathrm{i}s\sum_{i<j}\arg(\boldsymbol{x}_i-\boldsymbol{x}_j)} \quad \text{with} \quad s\in\mathbb{Z}. \tag{13.2.3}$$

This amounts to attaching $2s$ phase vortices to each electron[2]



Our task is now to find a many-body theory formulated in terms of this new degrees of freedom. In a second quantized version, Eq (13.2.3) looks like

$$c^\dagger(\boldsymbol{x}) \quad \mapsto \quad c^\dagger(\boldsymbol{x}) \exp\left[-2\mathrm{i}s \int d\boldsymbol{x}' \arg(\boldsymbol{x}-\boldsymbol{x}')\rho(\boldsymbol{x}')\right]. \tag{13.2.4}$$

Substituted into the Hamiltonian this leads to

$$H \quad \mapsto \quad \int d\boldsymbol{x}\, c^\dagger(\boldsymbol{x}) \left[\frac{1}{2m}\left(-\partial_{\boldsymbol{x}} + \hat{\boldsymbol{A}}(\boldsymbol{x})\right)^2 + V(\boldsymbol{x})\right] c(\boldsymbol{x}) + H_{\text{int}}[\rho], \tag{13.2.5}$$

where

$$\hat{\boldsymbol{A}}(\boldsymbol{x}) = \boldsymbol{A}_{\text{ext}}(\boldsymbol{x}) + \hat{\boldsymbol{a}}(\boldsymbol{x}) \quad \text{with} \quad \hat{\boldsymbol{a}}(\boldsymbol{x}) = -2s\int d\boldsymbol{x}' \, \frac{(x_1-x_1')\hat{\boldsymbol{x}}_1 + (x_2-x_2')\hat{\boldsymbol{x}}_2}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\rho(\boldsymbol{x}'). \tag{13.2.6}$$

This is very annoying! The kinetic energy operator became highly non-local and depends on six operators. Let us fix this. We can relocate the condition (13.2.6) to another place in the action. Two observations are needed for this:

(i) Eq. (13.2.6) is only giving rise to the transversal part of $\boldsymbol{A}$: $\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}_\perp$ as $\sum_i \partial_i \hat{a}_i = 0$.

(ii) $b = \epsilon^{ij}\partial_i a_{\perp,j}$ fulfills $b = -4\pi s \rho(\boldsymbol{x})$.

Using these two observations we can write

$$Z = \int D[\bar{\psi}\psi] D[a_\perp] D[\phi] e^{\mathrm{i}S_{\text{CF}}[\bar{\psi},\psi,a_\perp,\phi] + \mathrm{i}\frac{\Theta}{2}S'_{\text{CS}}[a_\perp,\phi]}, \tag{13.2.7}$$

where $\Theta = 1/2\pi s$. Furthermore,

$$S_{\text{CF}}[\bar{\psi},\psi,a_\perp,\phi] = \int d\boldsymbol{x} \int dt\, \bar{\psi}\left[\mathrm{i}\partial_t + \mu - \phi + \frac{1}{2m}\left(-\mathrm{i}\partial_{\boldsymbol{x}} + \hat{\boldsymbol{A}}\right)^2 - V\right]\psi + S_{\text{int}}[\bar{\psi},\psi]. \tag{13.2.8}$$

$$S'_{\text{CS}}[a_\perp,\phi] = -\int d\boldsymbol{x}\int dt\, \phi\, \underbrace{\epsilon_{ij}\partial_i a_{\perp,j}}_{b}. \tag{13.2.9}$$

$\hat{\boldsymbol{A}}$ is still given by $\boldsymbol{A}_{\text{ext}} + \hat{\boldsymbol{a}}$, but the constraint (13.2.6) is replaced by the functional δ-function

$$\int D[\phi] e^{\mathrm{i}\int d\boldsymbol{x}\int dt\, \phi\left(\frac{b}{4\pi s}+\rho\right)}. \tag{13.2.10}$$

[2]Cartoon due Kwon Park.

With this we are almost done. We see that $a_\perp = (\phi, \boldsymbol{a}_\perp)$ enters Z like a gauge field. However, $S'_{\rm CS}$ is not gauge invariant. Hence, we propose to use

$$S_{\rm CS}[a] = -\int dx^\mu \epsilon_{\mu\nu\sigma} a^\mu \partial_\nu a^\sigma. \tag{13.2.11}$$

with $x^\mu = (x^0, x^1, x^2)$; $\partial_\mu = (-\partial_0, \partial_1, \partial_2)$ which is gauge invariant. The old $S'_{\rm CS}$ is nothing but $S_{\rm CS}$ evaluated in the Coulomb gauge $\partial_\mu a^\mu = 0$. Therefore, our full effective theory is now given by

$$Z = \int D[\bar\psi\psi]D[a] \exp\left\{\mathrm{i}S_{\rm CF}[\bar\psi,\psi,a] + \mathrm{i}\frac{\Theta}{4}S_{\rm CS}[a]\right\}, \tag{13.2.12}$$

with

$$S_{\rm CF}[\bar\psi,\psi,a] = \int d\boldsymbol{x}\int dt\,\bar\psi\left[\mathrm{i}\partial_t + \mu - \phi + \frac{1}{2m}\left(-\mathrm{i}\partial_{\boldsymbol{x}} + \boldsymbol{A}_{\rm ext} - \boldsymbol{a}\right)^2 - V\right]\psi + S_{\rm int}[\bar\psi,\psi]. \tag{13.2.13}$$

13.2.2 Analyzing the composite fermion Chern-Simons theory

Before we embark on the analysis of the above effective theory, let us make a hand-waving mean-field analysis. We see that for $s = 1$, each electron binds two flux quanta. If we *assume* the density to be homogeneous (recall the plasma analogy for the Laughlin wave function), and if we neglect fluctuations, then the electrons see *on average* a flux corresponding to $\boldsymbol{A}_{\rm ext} - \langle\boldsymbol{a}\rangle$. In other words, the composite fermions see a smaller $\boldsymbol{B}$-field! Several scenarios are possible

(i) $\boldsymbol{A}_{\rm ext} = \langle\boldsymbol{a}\rangle \Rightarrow$ no magnetic field. This happens at $\nu = 1/2$. The fact that the composite fermion prediction at $\nu = 1/2$ looks like a Fermi liquid is one of the great successes of the composite fermion construction [1].

(ii) Maybe, for some filling fraction ν, the effective $\boldsymbol{B}$-field corresponding to $\boldsymbol{A}_{\rm ext} - \langle\boldsymbol{a}\rangle$ leads to an effective new filling fraction $\nu^* \in \mathbb{Z}$, i.e., the fractional quantum Hall effect for electrons would be mapped to an integer quantum Hall effect for composite fermions.

We are now trying to analyze the composite-fermion Chern-Simons (CF-CS) theory in mean-field. The only term which gives a real headache is the interaction term $S_{\rm int}[\bar\psi,\psi]$. We re-write it using a Hubbard-Stratanovich transformation

$$e^{\mathrm{i}S_{\rm int}} = \int D[\sigma]\exp\left\{\frac{\mathrm{i}}{2}\int dx^3 dx'^3\,\sigma(x)[V^{-1}](x,x')\delta(x_0 - x_0')\sigma(x') + \mathrm{i}\int dx^3(\rho(x) - \rho_0)\sigma(x)\right\}. \tag{13.2.14}$$

For the interpretation of the σ-field it helps to note that when completing the square, it appears as next to $\bar\psi\psi$, hence it describes a (rescaled) density field.[3] Now ψ and $\bar\psi$ (and $\rho = \bar\psi\psi$) only appear quadratically (linearly) in the action and we can integrate out $\bar\psi, \psi$. With this we obtain an effective theory

$$S_{\rm eff}[\sigma,a] = \underbrace{-\mathrm{i}\,\mathrm{tr}\log\left[\mathrm{i}\partial_0 + \mu - a_0 - \sigma + \frac{1}{2m}(-\mathrm{i}\nabla + A)^2\right]}_{S_\psi[a,A]} \tag{13.2.15}$$

$$- \rho_0\int dx^3\,\sigma(x) + \frac{1}{2}\int dx^3 dx'^3\,\sigma(x)[V^{-1}](x,x')\delta(x_0 - x_0')\sigma(x') \tag{13.2.16}$$

$$+ \frac{\Theta}{4}S_{\rm CS}[a], \tag{13.2.17}$$

where $A = A_{\rm ext} + a$. The first line arises from integrating out the fermions ψ. On this effective theory we want to apply a mean-field, or saddle-point, approximation. As there are no ψ-fields present anymore, it can be difficult to interpret the different terms in the theory. To

[3] We also say that we decouple the action in the density-density channel.

provide remedy to this problem, we note that the local density of fermions is given by taking the derivative of the original fermionic action with respect to $a_0(x)$. This property obviously survives the elimination of the ψ field. Therefore, we can get an "effective" expression for the density by

$$\frac{\delta S_\psi}{\delta a_0} = \rho[a, \sigma]. \tag{13.2.18}$$

Therefore,

$$\rho[a, \sigma] = \left[\mathrm{i}\partial_0 + \mu - a_0 - \sigma + \frac{1}{2m}(-\mathrm{i}\nabla + A)^2 \right]^{-1} (x, x). \tag{13.2.19}$$

Next, let us write down the saddle-point (Euler-Lagrange) equations. We start with

$$\left.\frac{\delta S_{\text{eff}}}{\delta a_o}\right|_{\bar{\sigma},\bar{a}} = 0 : \qquad \rho[\bar{a}, \bar{\sigma}] = \frac{1}{4\pi s}\bar{b}. \tag{13.2.20}$$

This is nothing but the expected relation between the $\bar{b}$ field and the density.[4]
Next, we also need to minimize the action with respect to the field σ

$$\frac{\delta S_{\text{eff}}}{\delta \sigma}|_{\bar{\sigma},\bar{a}} = 0 \qquad \Rightarrow \qquad \rho(x) - \rho_0 = -\int dx'^3 [V^{-1}](x, x')\sigma(x')\delta(x_0 - x_0'), \tag{13.2.21}$$

or

$$\sigma(x) = -\int dx'^3 V(x - x')[\rho(x') - \rho_0]\big|_{x_0'=x_0}. \tag{13.2.22}$$

Here we recognize that deviations of $\rho(x)$ from its mean value give rise to an "interaction potential" $\sigma(x)$. We can solve the mean-field equations by

$$\rho[\bar{a}, 0] = \rho_0 \tag{13.2.23}$$

$$\bar{\sigma} = \bar{a} = 0 \tag{13.2.24}$$

$$\bar{b} = 4\pi s \rho_0 \qquad \Rightarrow \boldsymbol{a} = 2s\nu \boldsymbol{A}_{\text{ext}}. \tag{13.2.25}$$

When can we expect this mean-field calculation to be reliable? Certainly, if the resulting ground-state is gapped, we can hope that fluctuations around the mean-field solutions will not do too much harm. One way to ensure a gapped mean-field solution is by asking for the effective $A - A_{\text{ext}} - a$ to give rise to a *filled effective Landau level.* Therefore we ask

$$\nu_{\text{eff}} = p \quad \text{or} \quad \Phi_{\text{eff}} = \frac{2\pi N}{p} \quad \text{with} \quad \Phi_{\text{eff}} = (B_{\text{ext}} - \bar{b})L^2. \tag{13.2.26}$$

Inserting $b = 4\pi s N/L^2$ we immediately obtain

$$\nu = \frac{2\pi N}{B_{\text{ext}}L^2} = \frac{2\pi N}{\frac{2\pi N}{p} + 4\pi s N} \qquad \Rightarrow \qquad \nu = \frac{p}{1 + 2sp}. \tag{13.2.27}$$

We can summarize the mean-field discussion with the following list and Fig. 13.1

(i) We can explain many fractions which are symmetrically distributed around $1/2s$ by an integer quantum Hall effect for composite fermions. Note, however, that the gap is entirely due to interactions!

(ii) For $\nu = p/2s$, CF-CS predicts a Fermi-liquid theory in $B_{\text{eff}} = 0$

  (a) This seems to describe $\nu = 1/2$ well [1].

  (b) For $3/2 = 1/2+1$ and $5/2 = 1/2+2$ one could have expected the same Fermi-liquid as they are nothing but the 1/2 plateaus in higher (real) Landau levels. This is however not the case. One can imagine that in these cases, residual interactions beyond the mean-field descriptions lead to an instability of the Fermi surface.

[4] Check that the minimization of the action with respect to a_1 and a_2 only provides the continuity equation of the density and does not give any further constraints in the mean-field value of a.

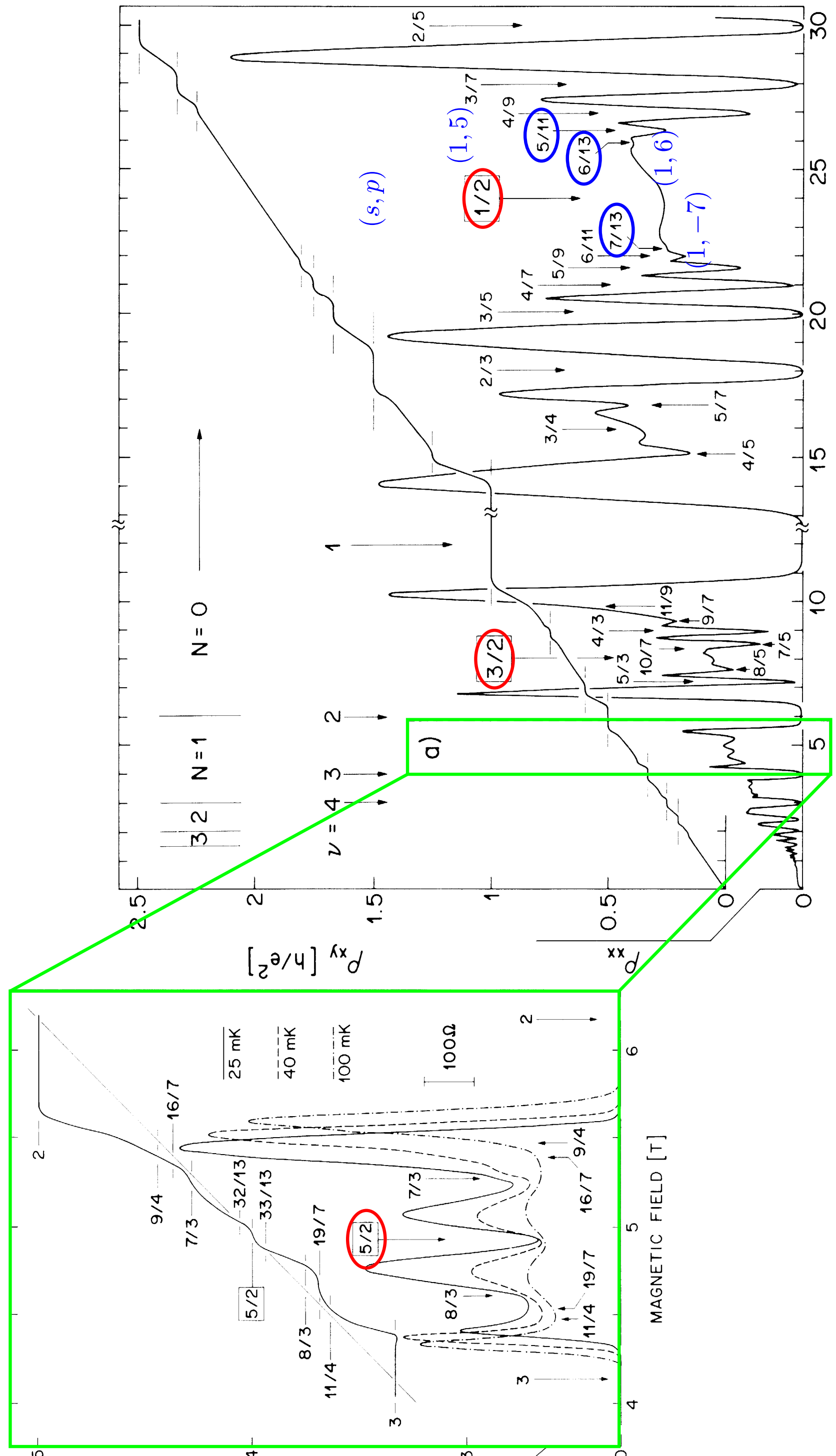


Figure 13.1: Overview of diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy}. The blue numbers denote the fractions which are well explained by an integer quantum Hall plateau of composite fermions. The inset shows the details around $\nu = 5/2$. Figure adapted from Ref. [2] (Copyright (1987) by The American Physical Society).

13.2.3 Fluctuations around the mean-field solution

We want to take a step beyond the mean-field considerations. For this, let us expand $S[a, A]$ to second order in a.[5] We could take the CF-CS action and expand to leading order around $\bar{a}$. However, we can do a much simpler thing. Let us just say that

$$S^{(2)}[a, A] = \frac{1}{2} \int dx^3 dx'^3 \, (A + a)^\mu(x) K_{\mu\nu}(x - x')(A + a)^\nu(x') + \frac{\Theta}{4} S_{\text{CS}}[a]. \tag{13.2.28}$$

Without actually calculating $K_{\mu\nu}$, we try to constrain it from general considerations

- $K_{\mu\nu}$ has to be gauge invariant.
- $K_{\mu\nu}(q)$ can be expanded in q.
- Via the Kubo formula (13.1.46), we know that $K_{\mu\nu}$ encodes the electromagnetic response.

We know that $\sigma_{11} = 0$ due to the gap for composite fermions. The transverse response, σ_{12}, however, can be non-zero. Recall, that

$$\sigma_{12} = -\mathrm{i} \lim_{\boldsymbol{q} \to 0} \frac{1}{\omega} K_{12}(\omega, \boldsymbol{q}). \tag{13.2.29}$$

From this we conclude that we have

$$K_{\mu\nu} = -\mathrm{i}\sigma_{12}^{(0)} \epsilon_{\mu\sigma\nu} q_\sigma. \tag{13.2.30}$$

Here $\sigma_{12}^{(0)}$ denotes the composite fermion mean-field value for the transverse response. Inserted into the expression for $S^{(2)}[a, A]$ we find

$$S^{(2)}[a, A] = \frac{\sigma_{12}^{(0)}}{2} S_{\text{CS}}[a + A] + \frac{\Theta}{4} S_{\text{CS}}[a]. \tag{13.2.31}$$

This effective action is clearly (i) gauge invariant, (ii) the lowest order expansion in q, and (iii) provides $K_{\mu\nu}$ that reproduces the electromagnetic of the effective theory. Actually, we would expect that

$$\frac{\delta^2 Z}{\delta A_\mu \delta A_\nu} \tag{13.2.32}$$

provides us with the desired response function. However, this is only true after we integrated out the fluctuations in a! What we need in the following is the formula valid for quadratic actions (Show!)

$$\int D[a] \, e^{c_1 S[a+b] + c_2 S[b]} = e^{\frac{1}{\frac{1}{c_1} + \frac{1}{c_2}} S[b]}. \tag{13.2.33}$$

Using this formula we obtain after integrating over the field a

$$S_{\text{eff}}[A] = \frac{1}{\frac{1}{\sigma_{12}^0} + \frac{2}{\Theta}} S_{\text{CS}}[A]. \tag{13.2.34}$$

And hence,

$$\sigma_{12} = \frac{e^2}{h} \frac{p}{1 + 2sp} \qquad s, p \in \mathbb{Z}. \tag{13.2.35}$$

[5] Why not in σ?

13.3 $\nu = 5/2$

We have seen that composite fermions allow us to extend the idea of the Laughlin wave function to many other fillings in a simple way. We also mentioned that the prediction of a Fermi-liquid at zero effective magnetic field was one of the great successes of the composite fermion approach. There is, however, even more interesting physics emerging from half-filled Landau levels.
In the presence of interactions, a Fermi surface is generically unstable towards symmetry breaking. At least at low enough temperatures. It seems, however, that for the lowest Landau level, the attached fluxes already account for all interaction effects (recall that the Laughlin wave-function is exact if we neglect pseudo-potentials v_s with $s > 1$).[6] A natural question that arises is what happens in higher Landau levels. Let us go through a chain of (somewhat handwaving) arguments

- For the lowest Landau level the flux attachment "exactly" accounts for interaction effects and renders the $\nu = 1/2$ state a gapless Fermi liquid.
- In the next Landau level, the free Hamiltonian is less optimized. Hence we can expect it to be slightly better tailored to satisfy the interaction effects.
- If we optimally absorbed the interaction effects in the lowest Landau level, we overdue in the second Landau level.
- To compensate for the over-accounted interactions, there is a residual *attraction* between composite fermions.

[6]Let us examine the two-particle problem on top of a Fermi surface

$$\left[-\frac{\hbar^2}{m}\nabla^2 + V(\boldsymbol{r})\right]\psi(\boldsymbol{r}) = \left(E + \frac{\hbar^2 k_{\mathrm{F}}^2}{m}\right)\psi(\boldsymbol{r}), \tag{13.3.1}$$

where $\boldsymbol{r}$ is the relative coordinate of the two-particle wave-function of a singlet of electrons. The energy E is measured with respect to the Fermi energy $\hbar^2 k_{\mathrm{F}}^2/2m$. We use the Fourier transform of the wave-function and the interaction potential

$$g(\boldsymbol{k}) = \int d\boldsymbol{r}\, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\psi(\boldsymbol{r}) \qquad \text{and} \qquad V(\boldsymbol{k}) = \int d\boldsymbol{r}\, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}V(\boldsymbol{r}). \tag{13.3.2}$$

Inserted into the above equation we obtain

$$\frac{\hbar^2 k^2}{m} g(\boldsymbol{k}) + \int \frac{d\boldsymbol{k}'}{(2\pi)^2} V(\boldsymbol{k}-\boldsymbol{k}') g(\boldsymbol{k}') = \left(E + \frac{\hbar^2 k_{\mathrm{F}}^2}{m}\right) g(\boldsymbol{k}). \tag{13.3.3}$$

If now assume $V(\boldsymbol{k})$ to be attractive with a strength $-V_0$ in a shell around the Fermi-surface we find

$$\left(-\frac{\hbar^2 k^2}{m} + E + \frac{\hbar^2 k_{\mathrm{F}}^2}{m}\right) g(\boldsymbol{k}) = -V_0 \int_{E_{\mathrm{F}} < (\hbar k')^2/2m < E_{\mathrm{F}}+\Lambda} d\boldsymbol{k}' g(\boldsymbol{k}'). \tag{13.3.4}$$

We can now divide by the bracket on the left and integrate over $\boldsymbol{k}$ in the shell around the Fermi-surface. With this we find the equation

$$1 = V_0 \int_{E_{\mathrm{F}} < (\hbar k')^2/2m < E_{\mathrm{F}}} \frac{d\boldsymbol{k}}{(2\pi)^2} \frac{1}{\frac{\hbar^2 k^2}{m} - E - \frac{\hbar^2 k_{\mathrm{F}}^2}{m}}. \tag{13.3.5}$$

We now use the constant density of states to go from a momentum integral over to an integral over the energy $\xi = \hbar^2 k^2/2m - E_{\mathrm{F}}$, where $N(0)$ is the density if states (at the Fermi level)

$$1 = V_0 N(0) \int_0^{\Lambda} d\xi \, \frac{1}{2\xi - E} = \frac{V_0 N(0)}{2} \log\left(-\frac{2\Lambda}{E}\right) \tag{13.3.6}$$

and therefore

$$E = -2\Lambda e^{-\frac{1}{V_0 N(0)}}. \tag{13.3.7}$$

In other words, two electrons bind into a Cooper pair [3]. Consequently, the Fermi-surface is unstable towards the formation of such pairs. It is the content of the Bardeen Cooper Schrieffer theory [4] to determine the resulting physics.

- Due to the strong magnetic field, it is profitable to form a superconductor out of spin-polarized composite fermions.
- The pairing has to be odd as the spin is symmetric.
- The resulting phase is a $p_x + \mathrm{i}p_y$ superconductor of composite fermions.
- We end up with a filling $1+1+1/2 = 5/2$. The two ones correspond to the two spin-species filling the lowest Landau level.
- We have seen in Chap. 6 that this state has non-abelian excitations in the form of Majorana zero modes bound to vortices.

It is important to note that due to the fact that we are dealing with a p-wave superconductor of *composite fermions* the resulting state and excitations are significantly more complicated than in the non-interacting case discussed in Chap. 6. In particular, $\nu = 5/2$ state is an intrinsically topologically ordered state with fractionally charged excitation with a charge $e^* = e/4$ [5]. The link between the full theory of the 5/2-state in terms of the famous Moore-Read state [6] and the p-wave superconductor interpretation was established by Read and Green [7]. In the framework of the superconductor language it was shown in a particularly transparent way that the quasiparticles exhibit non-abelian statistics by Ivanov [8].

References

1. Halperin, B. I., Lee, P. A. & Read, N. "Theory of the half-filled Landau level". *Phys. Rev. B* **47,** 7312. http://link.aps.org/doi/10.1103/PhysRevB.47.7312 (1993).
2. Willett, R. *et al.* "Observation of an even-denominator quantum number in the fractional quantum Hall effect". *Phys. Rev. Lett.* **59,** 1776. http://dx.doi.org/10.1103/PhysRevLett.59.1776 (1987).
3. Cooper, L. N. "Bound Electron Pairs in a Degenerate Fermi Gas". *Phys. Rev.* **104,** 1189. http://link.aps.org/doi/10.1103/PhysRev.104.1189 (1956).
4. Bardeen, J., Cooper, L. N. & Schrieffer, J. R. "Theory of Superconductivity". *Phys. Rev.* **108,** 1157. http://link.aps.org/doi/10.1103/PhysRev.108.1175 (1957).
5. Dolev, M., Heiblum, M., Umansky, V., Stern, A. & Mahalu, D. "Observation of a quarter of an electron charge at the $\nu = 5/2$ quantum Hall state". *Nature* **452,** 829. http://dx.doi.org/10.1038/nature06855 (2008).
6. Moore, G. & Read, N. "Nonabeloans in the fractional quantum Hall effect". *Nucl. Phys. B* **360,** 362. http://dx.doi.org/10.1016/0550-3213(91)90407-O (1991).
7. Read, N. & Green, D. "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect". *Phys. Rev. B* **61,** 10267. http://link.aps.org/doi/10.1103/PhysRevB.61.10267 (2000).
8. Ivanov, D. A. "Non-Abelian Statistics of Half-Quantum Vortices in p-Wave Superconductors". *Phys. Rev. Lett.* **86,** 268. http://link.aps.org/doi/10.1103/PhysRevLett.86.268 (2001).

Chapter 14

Summary

Learning goals

- We understand how all the models and systems discussed in the lecture can be systematized as part of the known types of topological phases.
- We understand the fundamental differences and the connections between noninteracting topological phases and intrinsic topological order.
- We can embed the contents of the lecture in the historical context of the field.

This chapter consists of two parts: We will first give a short historical account of topological condensed matter physics. Then, we will attempt to put the models and types of systems that we studied in this lecture in a systematic structure.

14.1 A historical recap of the lecture

This review is somewhat biased to experiments, since the ultimate relevance of physical insights lies in the connection to the real world, i.e., to experiments. There is much more theory work that often foreshadowed experimental developments.

Topology entered condensed matter physics in the form of topological defects. Specifically, Skyrmions (Tony Skyrme, 1962) were predicted as hypothetical particles, and much later confirmed in magnetic systems. Another topological defect – which found its experimental verification sooner – is a vortex in a superconductor or superfluid. It is at the basis of the Berezinskii-Kosterlitz-Thouless transition (1971/1973; Nobel prize), which is understood as a vortex-antivortex condensation in 2D (for instance in the XY spin model). The mechanism describes well 2D or quasi-2D (disordered) superconductors and Josephson junction arrays. Another early point where topological concepts entered condensed matter research is the study of defects in crystalline solids.

Then came the revolution of the quantum Hall effects, spearheaded by the integer quantum Hall effect (von Klitzing 1980; Nobel prize). They introduced quantized topological response functions, i.e., universally quantized observables that are expressed in constants of nature and do not depend on the system details. (Note that this is a stricter quantization constraint than in quantum mechanics, where for instance the energy quanta a system can absorb do depend on the system's details.) The mathematical understanding of the integer quantum Hall effect was soon delivered by Laughlin, Thouless, Kohmoto, Nightingale, Den Nijis (1892), defining the Chern number in a physical context by connecting it to the Hall conductivity $\sigma_{xy} = ne^2/h$, $n \in \mathbb{Z}$. In essentially the same system, the fractional quantum Hall effect was discovered (Laughlin, Störmer, Tsui 1982; Nobel prize) with Hall conductivities $\sigma_{xy} = \frac{p}{q}e^2/h$, with $p, q \in \mathbb{Z}$. This is a remarkable coincidence, as the integer and fractional effects are radically different from a theoretical perspective. The new paradigms coming with the fractional effect (intrinsic topological order and emergent quantum statistics) where therefore not understood until a few years later, with important contributions from X.-G. Wen at the beginning of the 1990s. There followed many studies that aimed to understand the cuprate high-temperature superconductors using topological physics in terms of anyons models and chiral spin liquids, but no experimental confirmation could be found.

In a somewhat parallel development, spin models were studied for their topological properties. Notably, Haldane conjectured that the spin-1 Heisenberg chains is gapped and supports topological spin-1/2 end excitations. An exactly soluble, but less natural model was conceived by Affleck, Kennedy, Lieb, Tasaki (AKLT model, 1987). The Haldane conjecture was experimentally confirmed in 1D spin chain materials in the 1990s.

While theoretical efforts in topological quantum matter diversified at the end of the 1990s, one important realization was the description of non-Abelian statistics of vortices in p-wave superfluids (Reed, Moore, Ivanov). All of this paved the way for Kitaev to write his most influential papers: on the honeycomb spin model and its reduction to the toric code as well as the work on Majorana chains. They proposed, respectively, the vision for topological software and topological hardware of a quantum computer. In particular, the Majorana wire was still a very abstract idea that was sharpened in several iterations by the community and eventually fueled the effort on Majorana fermions at Microsoft, with Leo Kouwenhoven and Charles Marcus as the lead experimentalists.

In parallel to researchers exploiting Kitaev's insights, 2005/06 marked the time where a revolution in topological band theory started, with the prediction of the QSHE by Bernevig-Hughes-Zhang and the experimental verification by Molenkamp as well as the theoretical discovery of $\mathbb{Z}_2$ topological insulators in 2D by Kane and Mele. In quick succession the 3D versions of the latter where predicted and found, setting in motion a back and forth between theoretical predictions and experimental verifications of new topological band structures that has not stopped since. Milestones are the discovery of Weyl semimetals in 2015 and the formalism of topological quantum chemistry in 2017. Furthermore, initiated by the work of Kane and Lubensky in 2013, researchers started to look for topological physics in classical systems, such as phononic, acoustic, or photonic metamaterials.

One influential name that has not been mentioned so far is Grigory E. Volovik, who's book "The universe in a helium droplet" anticipated much of the ideas described above. In addition, there has been a fruitful influx or exchange of ideas with high-energy and mathematical physicists over the years. For instance, Roman Jackiw published solutions to domain wall and vortex bound states (based on the Dirac equation) before they made an appearance in condensed matter. Juerg Froehlich, Leonard Susskind, Edward Witten brought important field-theoretical ideas, inspired often by quantum Hall physics where topological matter is the "cleanest".

14.2 A systematizing recap of the lecture

We close by organizing the lecture contents according to our current physical understanding of topological phases. Figure 14.1 attempts such a summary for gapped quantum phases at zero temperature. The biggest differentiation is between topological phases with or without intrinsic topological order: Those with intrinsic topological order *always* have a manifold-dependent topological ground state degeneracy. The others, so-called symmetry-protected topological phases (SPT) *always* gapless boundary modes if the boundary does not break the protecting symmetry, but never such a topological degeneracy in the ground state. (Note that some spatial symmetries give problems with the emergence of boundary modes, as any boundary may break them.). The only phases that do not quite fit in here are the integer quantum Hall phases (which exist both for fermions and for bosons). If we do not care about charge conservation (i.e., we are not interested in the quantized Hall conductivity), these phases are still topological with anomalous boundary modes that transport heat, but there is no symmetry needed to protect their topology.

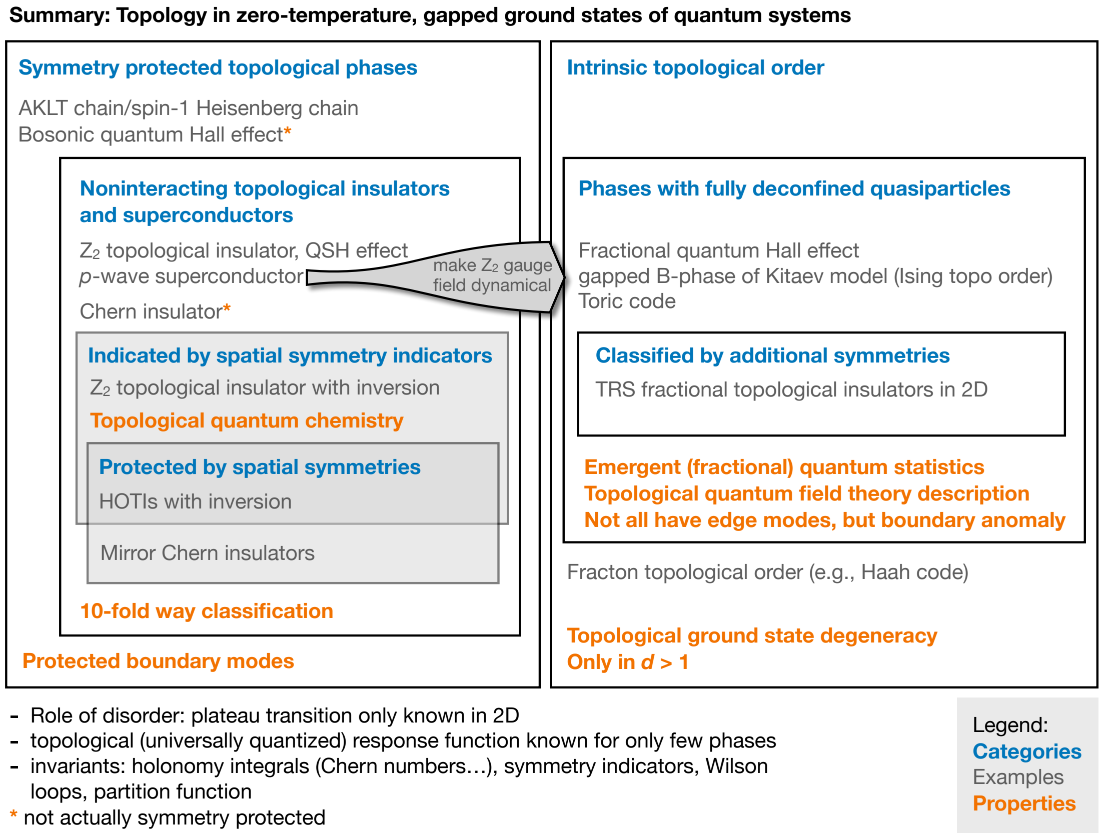


Figure 14.1: Overview of the lecture contents

Bosonic SPT phases always need interactions to be stabilized, as noninteracting Bosons fall into a boring Bose condensate ground state. For fermionic SPTs we know of examples that require interactions, but the ones we encountered in this lecture can all be defined for non-interacting systems. (The superconductors are somewhat of an exception – physically, they require interactions, but we considered only their mean-field BDG description which is effectively a non-interacting theory.) Foundational for the understanding of noninteracting fermionic SPTs is the 10-fold way. If spatial symmetries are added, their representation theory can be used to detect in which topological phase from the 10-fold way classification a system is (e.g., the inversion symmetry criterion for topological insulators), in which case we say the phase is symmetry-indicated. Furthermore, spatial symmetries can be used to protect new topological phases, such as a mirror Chern insulator. Some of these are indicated by representation theory invariants, but not all of them (for instance, in a C_n-symmetric TRS system, the mirror Chern number is symmetry indicated modulo n).
Turning to intrinsic topological order, we have to state a few things that have not been shown in the lecture. Intrinsic topological order exists for fermionic and bosonic systems in dimensions $d \geq 2$. In $d = 2$, the phases are fully classified and can always be understood from a topological quantum field theory and from their content of emergent quasiparticles (anyons) with fractional statistics. We have seen this at work in the FQHE and the toric code. In presence of additional symmetries such as charge conservation, fractional quantized response functions (e.g., fractional Hall conductivity) reflect the fractionalization. In $d > 2$ other types of intrinsic topological orders, so-called *fracton phases* exist, which also have topologically degenerate ground states but no quasiparticle picture can (fully) describe them.
It is important to realize that *all* intrinsic topological orders *are not* adiabatically connectable to a trivial phase, even if no symmetries are imposed. (Symmetries only offer a refinement of the

classification of intrinsic topological order phases among themselves.) This is in stark contrast to the SPTs: With the exception of the Hall effect type phases, *all* of them *are* adiabatically connected to a trivial phase if no symmetries are imposed.