Hadronic contributions to the muon anomalous magnetic moment from lattice QCD

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and the RBC/UKQCD collaborations
1 Introduction

2 Hadronic light-by-light (HLbL) scattering contribution
   - towards the continuum limit in finite volume
   - towards the infinite volume limit

3 Hadronic vacuum polarization (HVP) contribution

4 Beyond the Standard Model

5 Summary

6 References
Muon \( g-2 \) experimental measurement [Bennett et al., 2006]

E821 at BNL measured relative precession of muon spin to it's momentum

\[
\omega_a = \frac{g-2}{2} \frac{eB}{m} = a_\mu \frac{eB}{m}, \text{ the muon anomaly}
\]

The rate of detected electrons oscillates with \( \omega_a \), fit to

\[
N(t) = Be^{-\lambda t}(1 + A \cos \omega_a t + \phi)
\]

\( a_\mu \) (Expt) = 11 659 208.0(5.4)(3.3) \times 10^{-10} \quad 0.54 \text{ ppm!}
New muon g-2 experiments

Storage ring moved to FNAL for E989, begun in 2017

which is aiming for 0.14 ppm, $4\times$ improvement!

In Japan at J-PARC, the E34 experiment will measure $a_\mu$ using ultra-cold muons, different systematics ($\sim 2020$).
\[ \langle \mu(p') | J_\nu(0) | \mu(p) \rangle = -e \bar{u}(p') \left( F_1(q^2) \gamma_\nu + i \frac{F_2(q^2)}{4m} [\gamma_\nu, \gamma_\rho] q_\rho \right) u(p) \]

\[ a_\mu = F_2(0) \]
### Experiment - Theory

<table>
<thead>
<tr>
<th>SM Contribution</th>
<th>Value ± Error ($\times 10^{11}$)</th>
<th>Ref</th>
<th>Update</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED (5 loops)</td>
<td>116584718.951 ± 0.080</td>
<td>[Aoyama et al., 2012]</td>
<td></td>
</tr>
<tr>
<td>HVP LO</td>
<td>6923 ± 42</td>
<td>[Davier et al., 2011]</td>
<td>6926 (33) (Davier 16)</td>
</tr>
<tr>
<td></td>
<td>6949 ± 43</td>
<td>[Hagiwara et al., 2011]</td>
<td>6922 (25) (KNT17 preliminary)</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>−98.4 ± 0.7</td>
<td>[Hagiwara et al., 2011]</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>[Kurz et al., 2014]</td>
<td></td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>12.4 ± 0.1</td>
<td>[Kurz et al., 2014]</td>
<td></td>
</tr>
<tr>
<td>HLbL</td>
<td>105 ± 26</td>
<td>[Prades et al., 2009]</td>
<td></td>
</tr>
<tr>
<td>HLbL (NLO)</td>
<td>3 ± 2</td>
<td>[Colangelo et al., 2014a]</td>
<td></td>
</tr>
<tr>
<td>Weak (2 loops)</td>
<td>153.6 ± 1.0</td>
<td>[Gnendiger et al., 2013]</td>
<td></td>
</tr>
<tr>
<td>SM Tot (0.42 ppm)</td>
<td>116591802 ± 49</td>
<td>[Davier et al., 2011]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.43 ppm)</td>
<td>[Hagiwara et al., 2011]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.51 ppm)</td>
<td>[Aoyama et al., 2012]</td>
<td></td>
</tr>
<tr>
<td>Exp (0.54 ppm)</td>
<td>116592080 ± 63</td>
<td>[Bennett et al., 2006]</td>
<td></td>
</tr>
<tr>
<td>Diff (Exp − SM)</td>
<td>287 ± 80</td>
<td>[Davier et al., 2011] → 3.6σ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>261 ± 78</td>
<td>[Hagiwara et al., 2011] → 3.9σ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>249 ± 87</td>
<td>[Aoyama et al., 2012]</td>
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</tbody>
</table>

QCD errors largest, discrepancy large
New experiments + new theory = new physics?

- Fermilab E989 begins 2017, aims for 0.14 ppm
  J-PARC E34 $\sim$2020, aims for 0.3-0.4 ppm
  Today $a_\mu(\text{Expt}) - a_\mu(\text{SM}) \approx 2.9 - 3.6\sigma$ (possibly more)
- If both central values stay the same,
  - E989 ($\sim 4\times$ smaller error) $\rightarrow \sim 5\sigma$
  - E989 + new HLBL theory (models + lattice, 10%) $\rightarrow \sim 6\sigma$
  - E989 + new HLBL + new HVP (50% reduction) $\rightarrow \sim 8\sigma$
- Big discrepancy: new Physics $\sim 2\times$ Electroweak
- Lattice calculations important to trust theory errors, or lattice values may become (part of) the central values (muon g-2 theory initiative)
- Much progress, see many talks at Lattice 2017 (Granada) for latest results
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2. Hadronic light-by-light (HLbL) scattering contribution
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   - towards the infinite volume limit

3. Hadronic vacuum polarization (HVP) contribution

4. Beyond the Standard Model

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6. References
- **Models:** \((105 \pm 26) \times 10^{-11}\) \([\text{Prades et al., 2009, Benayoun et al., 2014}]\)
  \((116 \pm 40) \times 10^{-11}\) \([\text{Jegerlehner and Nyffeler, 2009}]\)

- **Model errors difficult to quantify** error now compatible with HVP error. see talk by A. Keshavarzi, muon g-2 theory initiative HVP working group workshop, Feb 2018, KEK

- **First lattice results promise reliable errors** \([\text{Blum et al., 2015, Blum et al., 2016, Blum et al., 2017a}]\) see also \([\text{Green et al., 2015, Asmussen et al., 2016}]\)

- **Dispersive/data approach also systematic**
  \([\text{Colangelo et al., 2014b, Pauk and Vanderhaeghen, 2014, Colangelo et al., 2015, Colangelo et al., 2017}]\)
The desired amplitude is obtained from a Euclidean space lattice calculation

\[ M_\nu(\vec{q}) = \lim_{t_{\text{src}} \to -\infty} e^{E_{q/2}(t_{\text{snk}} - t_{\text{src}})} \sum_{\vec{x}_{\text{snk}}, \vec{x}_{\text{src}}} e^{-i \frac{q}{2} \cdot (\vec{x}_{\text{snk}} + \vec{x}_{\text{src}})} e^{i \vec{q} \cdot \vec{x}_{\text{op}}} M_\nu(x_{\text{snk}}, x_{\text{op}}, x_{\text{src}}), \]

where

\[ -eM_\nu(x_{\text{src}}, x_{\text{op}}, x_{\text{snk}}) = \langle \mu(x_{\text{snk}}) J_\nu(x_{\text{op}}) \bar{\mu}(x_{\text{src}}) \rangle \]

\[ = -e \sum_{x,y,z} \sum_{x',y',z'} \mathcal{F}_\nu(x, y, z, x', y', z', x_{\text{op}}, x_{\text{snk}}, x_{\text{src}}). \]

and

\[
\left[ \left( \frac{-i \not{\!q}^+ + m_\mu}{2E_{q/2}} \right) \left( F_1(q^2) \gamma_\nu + i \frac{F_2(q^2)}{4m} [\gamma_\nu, \gamma_\rho] q_\rho \right) \left( \frac{-i \not{\!q}^- + m_\mu}{2E_{q/2}} \right) \right]_{\alpha \beta} = \left( M_\nu(\vec{q}) \right)_{\alpha \beta},
\]
Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

\[ \mathcal{F}^C_C (\vec{q}; x, y, z, x_{\text{op}}) = (-ie)^6 G^C_{\rho, \sigma, \kappa, \nu} (\vec{q}; x, y, z) \mathcal{H}^C_{\rho, \sigma, \kappa, \nu} (x, y, z, x_{\text{op}}) \]

\[ i^4 \mathcal{H}^C_{\rho, \sigma, \kappa, \nu} (x, y, z, x_{\text{op}}) \]

\[ = \sum_{q=u,d,s} \left( \frac{e_q/e}{6} \right)^4 \langle \text{tr} [ -i \gamma^\rho S_q (x, z) i \gamma^\kappa S_q (y, y') i \gamma^\sigma S_q (y, x_{\text{op}}) i \gamma^\nu S_q (x_{\text{op}}, x)] \rangle \rangle_{\text{QCD}} + 5 \text{ permutations} \]

\[ i^3 G^C_{\rho, \sigma, \kappa} (\vec{q}; x, y, z) \]

\[ = e^{\sqrt{m^2 + q^2 / 4(t_{\text{snk}} - t_{\text{src}})}} \sum_{x', y', z'} G_{\rho, \rho'} (x, x') G_{\sigma, \sigma'} (y, y') G_{\kappa, \kappa'} (z, z') \]

\[ \times \sum_{\vec{x}_{\text{snk}}, \vec{x}_{\text{src}}} e^{-i \vec{q} / 2 \cdot (\vec{x}_{\text{snk}} + \vec{x}_{\text{src}})} S (x_{\text{snk}}, x') i \gamma^\rho S (x', z') i \gamma^\kappa S (z', y') i \gamma^\sigma S (y', x_{\text{src}}) + 5 \text{ permutations} \]
Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

- The initial and final muon states are plane waves
- Do all sums in the QED part exactly (using FFT’s), QCD part done stochastically
- Key idea: contribution exponentially suppressed with $r = |x - y|$, so importance sample, concentrate on $r \lesssim \lambda_{\text{Compton}}$
- space-time translational invariance allows coordinates relative to the hadronic loop

$$\mathcal{M}_\nu(\vec{q}) = \sum_r \left\{ \sum_{z, \vec{x}_{op}} \mathcal{F}_\nu \left( \vec{q}, \frac{r}{2}, -\frac{r}{2}, z, \vec{x}_{op} \right) e^{i\vec{q} \cdot \vec{x}_{op}} \right\}$$

where $r = x - y$, $z \rightarrow z - w$, $x_{op} \rightarrow x_{op} - w$ and $w = (x + y)/2$
- We sum all the internal points over the entire space-time except we fix $x + y = 0$.
- $(x, y)$ pairs stochastically sampled, $z$ and $x_{op}$ sums exact
\[ \langle \mu(\vec{p}') | J_\nu(0) | \mu(\vec{p}) \rangle = -e \bar{u}(\vec{p}') \left( F_1(q^2) \gamma_\nu + i \frac{F_2(q^2)}{4m} [\gamma_\nu, \gamma_\rho] q_\rho \right) u(\vec{p}) \]

- implies \( F_2(0) \) only accessible by extrapolation \( q \to 0 \).
- Form is due to Ward Identity, or charge conservation
- need WI to be exact on each config, or error blows up as \( \vec{q} \to 0 \)
- To enforce WI compute average of diagrams with all possible insertions of \( J_\nu(x_{op}) \)
Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

WI allows a moment method that projects directly to \( q = 0 \)

\[
\mathcal{M}_\nu(\vec{q}) = \sum_{r, z, x_{\text{op}}} \mathcal{F}_\nu^C \left( \vec{q}, \frac{r}{2}, -\frac{r}{2}, z, x_{\text{op}} \right) \left( e^{i\vec{q} \cdot \vec{x}_{\text{op}}} - 1 \right)
\]

\[
\approx \sum_{r, z, x_{\text{op}}} \mathcal{F}_\nu^C \left( \vec{q}, \frac{r}{2}, -\frac{r}{2}, z, x_{\text{op}} \right) \left( i\vec{q} \cdot \vec{x}_{\text{op}} \right)
\]

\[
\left. \frac{\partial}{\partial q_i} \mathcal{M}_\nu(\vec{q}) \right|_{\vec{q}=0} = i \sum_{r, z, x_{\text{op}}} \mathcal{F}_\nu^C \left( \vec{q} = 0, r, -r, z, x_{\text{op}} \right) (x_{\text{op}})_i
\]
Sandwich $\mathcal{M}_\nu(\vec{q})$ between positive energy Dirac spinors $u(\vec{0}, s)$, $\bar{u}(\vec{0}, s)$

$$
\bar{u}(\vec{0}, s') \left( \frac{F_2(q^2 = 0)}{2m_\mu} \frac{i}{2} [\gamma_i, \gamma_j] \right) u(\vec{0}, s) = \bar{u}(\vec{0}, s') \frac{\partial}{\partial q_j} \mathcal{M}_i(\vec{q}) |_{\vec{q} = \vec{0}} u(\vec{0}, s)
$$

multiply both sides by $\frac{1}{2} \epsilon_{ijk}$, sum over $i$ and $j$,

$$
\frac{F_2(0)}{m} \bar{u}_{s'}(\vec{0}) \frac{\Sigma}{2} u_s(\vec{0}) = \sum_r \left[ \sum_{z, x_{op}} \frac{1}{2} \bar{x}_{op} \times \bar{u}_{s'}(\vec{0}) i F_C (\vec{0}; x = -\frac{r}{2}, y = +\frac{r}{2}, z, x_{op}) u_s(\vec{0}) \right]
$$

where $\Sigma_i = \frac{1}{4i} \epsilon_{ijk} [\gamma_j, \gamma_k]$. 
Lattice setup

- Photons: Feynman gauge, $\text{QED}_L$ [Hayakawa and Uno, 2008] (omit all modes with $\vec{q} = 0$)
- Gluons: Iwasaki gauge action (RG improved, plaquette+rectangle)
- Muons: $L_s = \infty$ free domain-wall fermions (DWF)
- Quarks: Möbius-DWF

$2+1f$ Möbius-DWF physical point ensembles (RBC/UKQCD) [Blum et al., 2014]

<table>
<thead>
<tr>
<th></th>
<th>$48^3 \times 96$</th>
<th>$64^3 \times 128$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^{-1}$ (GeV)</td>
<td>1.73</td>
<td>2.36</td>
</tr>
<tr>
<td>$a$ (fm)</td>
<td>0.114</td>
<td>0.084</td>
</tr>
<tr>
<td>$L$ (fm)</td>
<td>5.47</td>
<td>5.38</td>
</tr>
<tr>
<td>$L_s$</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>$m_\pi$ (MeV)</td>
<td>139</td>
<td>135</td>
</tr>
<tr>
<td>$m_\mu$ (MeV)</td>
<td>106</td>
<td>106</td>
</tr>
</tbody>
</table>
Test method in pure QED

QED systematics large, $O(a^4), O(1/L^2)$, but under control

Limits quite consistent with well known PT result
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Employ AMA with 2000 low-modes of the Dirac operator

Compute 2 point, 4 sequential $\times (112 + 256)$ sloppy propagators per configuration

On every configuration, do every distance $r \leq 5$ (and $r \leq 2$ twice), importance sample longer distances according to $\exp -0.01r/r^4$

choose location of 1st point randomly, second point according to above

(numbers summed over ensemble)
Physical point cHLbL contribution, $48^3$, 1.73 GeV lattice [Blum et al., 2017a]

- Measurements on 65 configurations, separated by 20 trajectories
- Ignore strange quark contribution (down by $1/17$ plus mass suppressed)
- Exponentially suppressed with distance
- Most of contribution by about 1 fm

$$a_{\mu}^{cHLbL} = 11.60 \pm 0.96 \times 10^{-10}$$
Physical point cHLbL contribution, $64^3$, 2.36 GeV lattice (preliminary)

- Measurements as before, but 43 configurations
- Exponentially suppressed with distance
- Most of contribution by about 1 fm
Disconnected contributions

SU(3) flavor:

Gluons within and connecting quark loops have not been drawn
To ensure loops are connected by gluons, explicit “vacuum” subtraction is required
Leading disconnected contribution

We use two point sources at \( y \) and \( z \), chosen randomly. The points sinks \( x_{\text{op}} \) and \( x \) are summed over exactly on lattice.

Only point source quark propagators are needed. We compute \( M \) point source propagators and all \( M^2 \) combinations are used to perform the stochastic sum over \( r = z - y \) (\( M^2 \) trick).

\[
\mathcal{F}^D_{\nu}(x, y, z, x_{\text{op}}) = (-ie)^6 G_{\rho, \sigma, \kappa}(x, y, z) \mathcal{H}^D_{\rho, \sigma, \kappa, \nu}(x, y, z, x_{\text{op}})
\]

\[
\mathcal{H}^D_{\rho, \sigma, \kappa, \nu}(x, y, z, x_{\text{op}}) = \left\langle \frac{1}{2} \Pi_{\nu, \kappa}(x_{\text{op}}, z) \left[ \Pi_{\rho, \sigma}(x, y) - \Pi_{\rho, \sigma}^{\text{avg}}(x - y) \right] \right\rangle_{\text{QCD}}
\]

\[
\Pi_{\rho, \sigma}(x, y) = -\sum_q (e_q/e)^2 \text{Tr}[\gamma_\rho S_q(x, y)\gamma_\sigma S_q(y, x)].
\]
Leading disconnected contribution

\[ \frac{F_{dHLbL}^D(0)}{m} \left( \sigma_{s',s} \right)_i = \sum_{r,x} \sum_{x_{op}} \frac{1}{2} \epsilon_{i,j,k} (x_{op})_j \cdot i \bar{u}_{s'}(\vec{0}) \mathcal{F}^D_k (x, y = r, z = 0, x_{op}) u_s(\vec{0}) \]

\[ \mathcal{H}^D_{\rho,\sigma,\kappa,\nu} (x, y, z, x_{op}) = \left\langle \frac{1}{2} \Pi_{\nu,\kappa} (x_{op}, z) \left[ \Pi_{\rho,\sigma} (x, y) - \Pi_{\rho,\sigma}^{avg} (x - y) \right] \right\rangle_{QCD} \]

\[ \sum_{x_{op}} \frac{1}{2} \epsilon_{i,j,k} (x_{op})_j \left\langle \Pi_{\rho,\sigma} (x_{op}, 0) \right\rangle_{QCD} = \sum_{x_{op}} \frac{1}{2} \epsilon_{i,j,k} (-x_{op})_j \left\langle \Pi_{\rho,\sigma} (-x_{op}, 0) \right\rangle_{QCD} = 0 \]

- Because of parity, the expectation value for the (moment of) left loop averages to zero.
- \([\Pi_{\rho,\sigma} (x, y) - \Pi_{\rho,\sigma}^{avg} (x - y)]\) is only a noise reduction technique. \(\Pi_{\rho,\sigma}^{avg} (x - y)\) should remain constant throughout the entire calculation.
Use AMA with 2000 low-modes of the Dirac operator and
randomly choose 256 “spheres” of radius 6 lattice units
Uniformly sample 4 (unique) points in each
do half as many strange quark props
Construct \((1024 + 512)^2\) point-pairs per configuration
Physical point dHLbL contribution, $48^3$, 1.73 GeV lattice [Blum et al., 2017a]

- strange contributes less than 5%

$$ a_{\mu}^{dHLbL} = -6.25 \pm 0.80 \times 10^{-10} $$

$$ a_{\mu}^{cHLbL} + a_{\mu}^{dHLbL} = 5.35 \pm 1.35 \times 10^{-10} $$
Physical point dHLbL contribution, $64^3$, 2.36 GeV lattice (preliminary)

- 39 configurations

![Graph 1](image1.png)

![Graph 2](image2.png)
cHLBL, dHLbL: lattice spacing effect (preliminary)

- Effects tend to cancel
- Linear in $a^2 \rightarrow 0$ extrapolation
- Collecting more statistics
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Mainz group made first concrete proposal for $\text{QED}_\infty$

$\text{QED}_\infty$: muon, photons computed in infinite volume (c.f. HVP)

QCD mass gap: $\mathcal{H}_{\rho,\sigma,\kappa,\nu}^C(x, y, z, x_{\text{op}}) \sim \exp -m_\pi \times \text{dist}(x, y, z, x_{\text{op}})$

QED weight function does not grow exponentially

Then leading FV error is exponentially suppressed (c.f. HVP) instead of $O(1/L^2)$
QED\(_\infty\) weighting function [Blum et al., 2017b]

\[
\mathcal{G}_{\rho, \sigma, \kappa}(x, y, z) + 5 \text{ perms.}
\]

- Note Hermitian part gives same \( F_2 \) but is infrared finite,

\[
\mathcal{G}^{(1)}_{\rho, \sigma, \kappa}(x, y, z) = \frac{1}{2} \mathcal{G}_{\rho, \sigma, \kappa}(x, y, z) + \frac{1}{2} \mathcal{G}_{\rho, \sigma, \kappa}(x, y, z)^\dagger
\]

- In units of the muon mass \( m_\mu \),

\[
\mathcal{G}^{(1)}_{\sigma, \kappa, \rho}(y, z, x) = \frac{\gamma_0 + 1}{2} i\gamma_\sigma (-\partial_y + \gamma_0 + 1)i\gamma_\kappa (\partial_x + \gamma_0 + 1)i\gamma_\rho \frac{\gamma_0 + 1}{2} \\
\times \frac{1}{4\pi^2} \int d^4 \eta \frac{1}{(\eta - z)^2} f(\eta - y)f(x - \eta)
\]
Current conservation implies $\sum_x H_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{op}) = 0 \ (V \to \infty \ \text{and} \ a \to 0)$.

Subtract terms that vanish as $a, V \to 0$

$$G^{(2)}_{\rho,\sigma,\kappa}(x, y, z) = G^{(1)}_{\rho,\sigma,\kappa}(x, y, z) - G^{(1)}_{\rho,\sigma,\kappa}(y, y, z) - G^{(1)}_{\rho,\sigma,\kappa}(x, y, y) + G^{(1)}_{\rho,\sigma,\kappa}(y, y, y)$$

Subtraction changes (may reduce) $a$ and $V$ systematic errors (c.f. HVP)

Further, $G^{(2)}_{\rho,\sigma,\kappa}(z, z, x) = 0$ so short distance $O(a^2)$ effects suppressed.

The 4-dim integral is (pre-)calculated numerically with CUBA library (cubature rules).

Translation/rotation symmetry: parametrize $(x, y, z)$ by 5 parameters on $N^5$ grid points (Mainz uses 3 params by averaging over muon time direction).

(linearly) Interpolate grid in stochastic integral over $(x, y)$
QED$_\infty$ results- pure QED, interpolation error \cite{Blum et al., 2017b}

- $N = 6, 8, 10, 12, 14, 16 \to \infty$ (2nd order in $1/N^2$ fits)

\begin{align*}
\mathcal{G}^{(1)}_{\rho, \sigma, \kappa} (x, y, z) \\
\mathcal{G}^{(2)}_{\rho, \sigma, \kappa} (x, y, z)
\end{align*}

\begin{itemize}
  \item $m_L = 3.2; ma = 0.066667$
  \item $m_L = 3.2; ma = 0.100000$
  \item $m_L = 4.8; ma = 0.100000$
  \item $m_L = 9.6; ma = 0.200000$
\end{itemize}

Taku Izubuchi, Lattice 2017, June 23, 2017
QED\(_\infty\) results- pure QED, lattice-spacing error [Blum et al., 2017b]

- lattice spacing error \(\approx\)\(\text{const}\) for \(mL \gtrsim 4.8\)
- FV effect \(\lesssim 1\%\) for \(mL = 9.6\)
- fit: \(F_2(L, a) = F_2(L) + k_1 a^2 + k_2 a^4\)

\[ mL = 3.2 \quad mL = 4.8 \quad mL = 6.4 \quad mL = 9.6 \]

\[
\begin{align*}
G^{(1)}_{\rho,\sigma,\kappa}(x, y, z) & \\
G^{(2)}_{\rho,\sigma,\kappa}(x, y, z) & 
\end{align*}
\]
Take $F_2(\infty) \approx F_2(mL = 9.6)$

- results for $m_{\text{loop}} = m_{\text{line}} (a_e)$
- Similar results hold for $m_{\text{loop}} = 2m_{\text{line}}$
- $F_2/(\alpha/\pi)^3 = 0.3686(37)(35)$ and $0.1232(30)(28)$ compared to QED perturbation theory results: $0.371$ and $0.120$
QED$_\infty$ + QCD preliminary results: cHLbL up to 2.5 fm

$24^3$ID, $48^3$ID (1 GeV); $48^3$I (1.73 GeV); $64^3$I (2.36 GeV)

- different QED weights, agree if summed over all points (up to $O(a^2, e^{-L})$)
- same stochastic sampling as before for hadronic loop
QED$_{\infty}$+QCD preliminary results: chLbL up to 8 fm

$24^3$ID, $48^3$ID (1 GeV); $48^3$I (1.73 GeV); $64^3$I (2.36 GeV)

- Subtraction also gives smaller statistical errors
- QED$_{\infty}$ has much larger statistical errors than QED$_{L}$

\begin{align*}
F_2(0)/(\alpha/\pi)^3 & \\
\text{R}_{\text{min}} & = 2.5 \text{ fm}
\end{align*}
\[ \text{QED}_\infty + \pi^0 \text{-pole; Mainz group} \text{ talk by N. Asmussen, Lattice 2017} \]

\[ a_{Hbl}(|y|_{\text{max}}) \]

- \( \infty \)-volume model, cut off at finite distance \(|y|_{\text{max}}\)
- similar QED\(_\infty\), different FV effects (no sub, ...)

\( m_\pi = 300 \text{ MeV} \)
\( m_\pi = 600 \text{ MeV} \)
\( m_\pi = 900 \text{ MeV} \)
\[ a^{-1} = 1.0, 1.4 \text{ GeV}, \text{ physical mass ensembles (I-DSDR gauge field action)} \]

- 24\(^3\), 32\(^3\) and 48\(^3\) lattices
- physical sizes of 4.8, 6.4, and 9.6 fm
- use QED\(_L\), QED\(_\infty\) to extrapolate HLbL g-2 to QCD\(_\infty\)
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HVP contribution to muon g-2 [Blum, 2003, Lautrup et al., 1971]

Using lattice QCD and continuum, $\infty$-volume pQED

$$a_\mu(\text{HVP}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 f(q^2) \hat{\Pi}(q^2)$$

$f(q^2)$ is known, $\hat{\Pi}(q^2)$ is subtracted HVP, $\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$, computed directly on Euclidean space-time lattice

$$\Pi^{\mu\nu}(q) = \int e^{iqx} \langle j^\mu(x)j^{\nu}(0) \rangle \quad j^\mu(x) = \sum_i Q_i \bar{\psi}(x) \gamma^\mu \psi(x)$$

$$= \Pi(q^2)(q^\mu q^{\nu} - q^2 \delta^{\mu\nu})$$
\[
\Pi(q^2) - \Pi(0) = \sum_t \left( \frac{\cos qt - 1}{q^2} + \frac{1}{2} t^2 \right) C(t)
\]

\[
C(t) = \frac{1}{3} \sum_{x, i} \langle j_i(x) j_i(0) \rangle
\]

\[
a^\text{HVP}_\mu = \sum_t w(t) C(t)
\]

\[
w(t) = 2 \int_0^\infty \frac{d\omega}{\omega} f(\omega^2) \left[ \frac{\cos \omega t - 1}{(2 \sin \omega t/2)^2 + t^2} + \frac{t^2}{2} \right]
\]

\(w(t)\) includes the continuum QED part of the diagram
HVP contributions computed on lattice (u,d,s,c quarks)

Quark-connected piece with by far dominant part from up and down quark loops,
\[ \mathcal{O}(700 \times 10^{-10}) \]

Quark-disconnected piece,
\[ -9.6(4.0) \times 10^{-10} \]

QED corrections,
\[ \mathcal{O}(10 \times 10^{-10}) \]

QED and strong isospin corrections

![Diagrams](image-url)

- **QED**
  - Focus on $V$, $S$, $F$, and $M$ so far

- **Strong Isospin**
small statistical errors from full volume 2000 low-mode average and improved stochastic method (disc)

BMW confirms our $a^\text{DISC}_\mu$ result

statistical errors grow at long distance
QED and strong isospin corrections

RBC/UKQCD, [Blum et al., 2018]

- Importance sampling a’la HLbL calc
- physical pion mass, $a^{-1} = 1.73$ GeV lattice
- increased statistics in progress
Connection to the R-ratio  Bernecker-Meyer 2011

\[ \Pi(-Q^2) = \frac{1}{\pi} \int_0^\infty ds \frac{s}{s + Q^2} \sigma(s, e^+ e^- \rightarrow \text{had}) \]

\[ R(s) = \frac{\sigma(s, e^+ e^- \rightarrow \text{had})}{\sigma(s, e^+ e^- \rightarrow \mu^+ \mu^-, \text{tree})} = \frac{3s}{4\pi\alpha^2} \sigma(s, e^+ e^- \rightarrow \text{had}) \]

Fourier transform gives

\[ C(t) \propto \int_0^\infty d(\sqrt{s})R(s)s e^{-\sqrt{s}t} \]
Lattice - R-ratio

Jegerlehner 2016 comparison
Bernecker-Meyer 2011

Comparison of \( R \)-ratio

Light+Strange (64I) vs. R-ratio

\( t \) / fm

\( x \times 10^{-10} \)

RBC/UKQCD [Blum et al., 2018]

Lattice more accurate at intermediate distances
In this section we expand on a selection of technical details ... in particular also with regard to the up and down quark connected contribution in the isospin limit.

At sufficiently long distances, we have reduced contributions from high-energy scales. The top panel of Fig. 9 shows the corresponding contributions to short-distance and long-distance projections of the weighted correlator $C(t)_{\text{w}}$. We notice that, as expected, $C(t)_{\text{w}}$ is shown in Fig. 9.

The continuum limit of a selected property of the domain-wall discretization used in this work is consistent with a naive power-counting estimate. We attribute this mild continuum limit to the fact of light-quark window contributions being available on different lattice data may facilitate cross-checks between different choices of window.

For this estimator, we find that we are able to saturate the improved statistical estimator including a full low-mode calculation. This is left to future work.

In Tabs. S I-VII we provide results for different values of $t$ and window.

Columns $C(t)_{\text{w}}$, $C(t)_{\text{w}} \theta(t,1.5\text{fm},0.15\text{fm})$, $C(t)_{\text{w}} [1-\theta(t,0.4\text{fm},0.15\text{fm})]$}

Select window in $t$ (or, $\equiv \sqrt{s}$)
Window method

$$a_{\mu} = \sum_{t} w_{t} C(t) = a^{SD}_{\mu} + a^{W}_{\mu} + a^{LD}_{\mu}$$

$$a^{SD}_{\mu} = \sum_{t} w_{t} C(t) [1 - \Theta(t, t_{0}, \Delta)]$$

$$a^{W}_{\mu} = \sum_{t} w_{t} C(t) [\Theta(t, t_{0}, \Delta) - \Theta(t, t_{1}, \Delta)]$$

$$a^{LD}_{\mu} = \sum_{t} w_{t} C(t) \Theta(t, t_{1}, \Delta)$$

where $$\Theta(t, t', \Delta) = \frac{[1 + \tanh((t - t')/\Delta)]}{2}$$

Take $$a^{W}_{\mu}$$ from lattice, rest from R-ratio
The innermost error-bar corresponds to the statistical uncertainty. This is the currently most precise determination of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. We have presented both a complete first-principles calculation of the window method with lattice statistical, lattice systematic, R-ratio systematic errors given separately. For the former we find the first error is statistical and the second is systematic. For the latter we find a combination with R-ratio data. For the former we find a combination with R-ratio data.

The references are ETMC 2013 [41], HPQCD 2016 [42], Mainz 2017 [610], BMW 2017 [39], HLMNT 2011 [4], DHMZ 2012 [44], Jegerlehner 2017 [5]. No new physics [3].

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Combined lattice (u,d,s,c) and R-ratio result for $a_\mu$ RBC/UKQCD [Blum et al., 2018]
Summary of HVP theory results

The plot shows the summary of HVP theory results, with the latest RBC/UKQCD results [Blum et al., 2018] marked. The data points correspond to different collaborations and are shown in various colors. The x-axis represents the muon anomalous magnetic moment ($a_\mu \times 10^{10}$). The graph includes contributions from lattice QCD, R-ratio, and other methods, with references to various studies such as ETMC 2013, HPQCD 2016, Mainz 2017, BMW 2017, RBC/UKQCD 2018, HLMNT 2011, DHMZ 2012, DHMZ 2017, Jegerlehner 2017, and No new physics. The uncertainty bands and data points indicate the statistical and systematic errors. The plot also highlights the need for further computations to improve the accuracy of the predictions.

Latest RBC/UKQCD results [Blum et al., 2018]
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Explaining g-2 beyond the SM

If there really is a discrepancy, where does it come from?

Most likely scenario is still SUSY (?) [Bach et al., 2015, Athron et al., 2016, Belyaev et al., 2016], . . .

SUSY signatures at LHC

But there are other models too: 2HDM [Crivellin et al., 2016, Cherchiglia et al., 2016], Dark Matter [Kobakhidze et al., 2016], . . ., LFV [Altmannshofer et al., 2016], light scalars [Batell et al., 2017], . . .
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Hadronic Light-by-Light

- Lattice QCD(+QED) calculations done with physical masses, large boxes + improved measurement algorithms
- Physical point calculations complete at $a = 0.114$ fm [Blum et al., 2017a]
- Physical point nearly complete at $a = 0.084$ fm (increasing statistics)
- together, good control of non-zero $a$ systematic error.
- FV corrections: $\text{QED}_\infty + \text{large 9.5 fm QCD box}$ (underway)
- Need non-leading disconnected diagrams
- Lattice: unlikely that HLbL contribution will rescue standard model
Hadronic vacuum polarization

- Many groups working on it (see Lattice 2017 talks—world-wide effort!)
- Lattice QCD(+QED) calculations done with physical masses, large boxes + improved measurement algorithms
- Disconnected contributions computed by 2 groups
- Included NLO QED and Strong Isospin breaking corrections
- Lattice and R-ratio results cross-checked and combined with window method
- Window method allows further error reduction over R-ratio alone
  
  most accurate determination to date RBC/UKQCD [Blum et al., 2018]

On track for solid SM result in time for E989 result
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