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Classical Integrability and the Heisenberg Spin Chain

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Abstract

We study the isotropic one dimensional Heisenberg spin chain, focusing on its integrability. As a starting point we motivate the Heisenberg Hamiltonian. The main objective of the report is demonstrating the model's integrability. This is accomplished by constructing a family of commuting operators in which the Hamiltonian is contained. In order to provide some motivation for the construction employed, we briefly review Liouville integrability of classical systems and its Lax pair formalism.

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1 The Heisenberg model

This report is based on the presentation about *Classical Integrability and the Heisenberg Spin Chain*, given in the Spring Semester of 2018 at ETH Zürich as part of the Proseminar in *Algebra, Topology, and Group Theory in Physics*.

The Heisenberg model of an isotropic one dimensional spin chain describes a chain of N -electron sites. Consequently, the Hilbert space \mathcal{H} consists of the tensor product space of all individual quantum spin spaces, where we identify each quantum spin space with \mathbb{C}^2 .

$$\mathcal{H} = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$$

The Hamiltonian is given by the sum of all nearest neighbour interactions,

$$H = -J \sum_{i=1}^N \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} - \frac{1}{4} \mathbb{I} \right).$$

The Heisenberg model is a simple model which gives insight to ferromagnetic and anti-ferromagnetic properties of materials. The structure of the Heisenberg model will be motivated by taking a look at the exchange interaction occurring for identical particles. An important aspect is that it is an integrable model, which allows for a solution even in the thermodynamical limit of $N \rightarrow \infty$. Rather than discussing explicit results, we focus on the integrability of the Heisenberg model. In section 2.1 the notion of Liouville integrability is first introduced in its classical understanding. By looking at the special case of Lax pairs we will motivate the mathematical steps needed in order to show that the Heisenberg model is integrable. In section 3 the integrability of the Heisenberg model is explicitly shown. Applying the Lax operator formalism we will construct a monodromy matrix and extract a family of $N - 1$ commuting operators. By showing that the Hamiltonian itself is part of this family we show that the $N - 1$ operators are also conserved. In order to fulfil the criteria of integrability we will enlarge the $N - 1$ operators to a family of N commuting and conserved operators by adding one component of the total spin.

1.1 Motivation of the Heisenberg model

We motivate this section by following the thesis *Heisenberg Model, Bethe Ansatz and Random Walks* by Lenhard L. Ng [3].

We consider a one dimensional crystal consisting of N identical atoms. Each atom

has, apart from closed shells, only one conduction electron. In general this is a complicated many-body structure. There is an internal structure of each atom within the closed shells and there are interactions between all electrons.

Our goal is to approximate this complicated system via a simpler spin-spin interaction. This is done by only focusing on the single conduction electron of each atom. We lose all information about the internal structure of each atom, but keep the part that is relevant for the description of (anti-) ferromagnetism.

1.1.1 Exchange interaction

We start of by looking at the exchange interaction. It is important to note that the exchange interaction only occurs between identical particles. As an example we consider a two particle system. Since both particles are identical we can write the total wave function of the system in a symmetric and antisymmetric way,

$$\begin{aligned}\Psi_S(\vec{r}_1, \vec{r}_2) &= \frac{1}{\sqrt{2}}[\phi_1(\vec{r}_1)\phi_2(\vec{r}_2) + \phi_2(\vec{r}_1)\phi_1(\vec{r}_2)], \\ \Psi_A(\vec{r}_1, \vec{r}_2) &= \frac{1}{\sqrt{2}}[\phi_1(\vec{r}_1)\phi_2(\vec{r}_2) - \phi_2(\vec{r}_1)\phi_1(\vec{r}_2)].\end{aligned}$$

Here $\phi_i(\cdot)$ denotes the single particle wave function and the symmetry refers to the exchange of particle labels e.g. identifying particle 1 with particle 2 and vis versa.

The symmetry of the wave function in position space has an effect on the expectation value of the Hamiltonian,

$$\begin{aligned}E_S &= \langle \Psi_S | H | \Psi_S \rangle = \iint \Psi_S^*(\vec{r}_1, \vec{r}_2) \hat{H} \Psi_S(\vec{r}_1, \vec{r}_2) d^3r_1 d^3r_2, \\ E_A &= \langle \Psi_A | H | \Psi_A \rangle = \iint \Psi_A^*(\vec{r}_1, \vec{r}_2) \hat{H} \Psi_A(\vec{r}_1, \vec{r}_2) d^3r_1 d^3r_2.\end{aligned}$$

By explicitly plugging in the expressions for $\Psi_S(\vec{r}_1, \vec{r}_2)$ and $\Psi_A(\vec{r}_1, \vec{r}_2)$ we can compute the difference of the two expectation values,

$$E_S - E_A = 2 \langle \phi_1(\vec{r}_1)\phi_2(\vec{r}_2) | \hat{H} | \phi_2(\vec{r}_1)\phi_1(\vec{r}_2) \rangle. \quad (1)$$

The energy thus depends on the symmetry of the total wave function and the difference in energy is proportional to the exchange interaction

$$J_{ij} = \langle \phi_i(\vec{r}_i)\phi_j(\vec{r}_j) | \hat{H} | \phi_j(\vec{r}_i)\phi_i(\vec{r}_j) \rangle. \quad (2)$$

The next step is to approximate the complicated Hamiltonian acting on position space as an effective Hamiltonian acting only on spin space. This can be done by only considering the single conduction electrons of each atom. Due to the Pauli exclusion principle they will determine the symmetry of the wave function.

1.1.2 Two electron system

In a system made up of two electrons each electron has a position in \mathbb{R}^3 and a spin in $\mathfrak{h}_j \cong \mathbb{C}^2$. So the Hilbert space of the total system is given by the tensor product of a space component and a spin component. This means that the total wave function of the system can be written as the tensor product $\Psi_{space} \otimes \Psi_{spin}$.

The Pauli exclusion principle states that the total wave function of a system consisting of electrons must be antisymmetric. Therefore the symmetry of the spin component of the wave function determines the symmetry of the space component of the wave function, which in turn determines the energy of our effective Hamiltonian. The symmetry of Ψ_{spin} is determined by the expectation value of $\mathbf{S}_1 \cdot \mathbf{S}_2$. This is shown by writing down all symmetric and antisymmetric spin wave functions.

$$\begin{aligned}\Psi_{spin}^S &\sim |\uparrow\uparrow\rangle; |\downarrow\downarrow\rangle; |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ \Psi_{spin}^A &\sim |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\end{aligned}$$

By rewriting the operator $\mathbf{S}_1 \cdot \mathbf{S}_2 = 1/2(S_1^+ S_2^- + S_1^- S_2^+) + S_1^z S_2^z$, we find that both the symmetric spin wave functions and the antisymmetric spin wave function are eigenvectors of the operator $\mathbf{S}_1 \cdot \mathbf{S}_2$ but with different eigenvalues.

$$\begin{aligned}\Psi_{spin}^S &\text{ eigenvector of } \mathbf{S}_1 \cdot \mathbf{S}_2 \text{ with eigenvalue } \frac{\hbar^2}{4}, \\ \Psi_{spin}^A &\text{ eigenvector of } \mathbf{S}_1 \cdot \mathbf{S}_2 \text{ with eigenvalue } -\frac{3\hbar^2}{4}.\end{aligned}$$

We can thus indeed differentiate between symmetric and antisymmetric spin wave functions via the operator $\mathbf{S}_1 \cdot \mathbf{S}_2$.

Now we can explicitly construct an effective Hamiltonian acting on the spin space which will formally be equivalent to the Hamiltonian acting on position space,

$$H_{spin} = -2J_{12}\mathbf{S}_1 \cdot \mathbf{S}_2 - \frac{1}{2}\mathbb{I}. \quad (3)$$

The formal equivalence to H_{space} comes from the fact that when we act with H_{spin} on one of our spin wave functions we find

$$H_{spin} \Psi_{spin}^S = E'_A \Psi_{spin}^S, \quad H_{spin} \Psi_{spin}^A = E'_S \Psi_{spin}^A,$$

which is equivalent up to a constant shift to the energy we find when acting with H_{space} on the position wave function with the opposite symmetry to the spin wave function. This Hamiltonian can be generalised to N -electrons by taking the sum of two-particle interactions.

1.1.3 Heisenberg Hamiltonian

The exchange interaction is proportional to the overlap of the two single wave functions of the involved electrons. This means that the further apart two electrons are, the smaller is the exchange interaction. Since this falloff is very rapid and we are considering a crystal with localised electrons, we can make the further approximation that only nearest neighbours interact

$$J_{ij} = \begin{cases} J, & \text{i,j nearest neighbor} \\ 0, & \text{else.} \end{cases}$$

Therefore the Heisenberg Hamiltonian for an isotropic one dimensional spin chain with N sites is given by summing over all nearest neighbour sites.

$$H = -J \sum_{i=1}^N \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} - \frac{1}{4} \mathbb{I} \right). \quad (4)$$

Here the periodic boundary condition $\mathbf{S}_{N+1} = \mathbf{S}_1$ is implied.

2 Classical integrability

As a preparation for the Heisenberg spin chain, we discuss integrability in classical systems. This section is adapted from the book *Introduction to Classical Integrable Systems* by O. Babelon, D. Bernard and M. Talon [4].

In the context of classical mechanics, only a few exact solutions of Newton's equations of motion are known. A famous example is the Kepler problem which was solved by Newton himself.

In the nineteenth century, Liouville provided a general procedure to determine if the equations of motion of a given system are "solvable by quadratures". This is known as Liouville integrability.

2.1 Liouville integrability

In classical mechanics, the Hamiltonian of a system is a function on the phase space $H(p_i, q_i)$, where q_i denotes the position coordinate on the phase space and p_i denotes the momentum. The equations of motion for a system are given by a first order differential system,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (5)$$

where the dot refers to the time derivative. This implies that the time derivative of any function on the phase space $F(p, q)$ can be obtained by

$$\dot{F} = \{H, F\}, \quad (6)$$

where the Poisson bracket of two functions F and G is defined as

$$\{F, G\} \equiv \sum_i \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i}. \quad (7)$$

The Poisson bracket for the phase space coordinates p_i and q_i are given by

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}. \quad (8)$$

In some cases, eqs. (5) can be solved exactly for a given dynamical system. These dynamical systems are called Liouville integrable systems.

Definition. A dynamical system on a phase space of dimension $2n$ is Liouville integrable if one can find n independent functions F_i on the phase space such that

$$H = F_1, \quad \{F_i, F_j\} = 0.$$

The solution of the equation of motion of a Liouville integrable system can be obtained by “quadratures”, i.e. the solution can be expressed in terms of integrals.

2.2 Lax pair

For Liouville integrable systems there is a special way of writing the Hamilton evolution equations (5) via a Lax pair,

$$\frac{dL}{dt} \equiv \dot{L} = [M, L]. \quad (9)$$

Where L and M are two matrices and $[M, L] = ML - LM$ denotes the commutator of the matrices M and L .

Example 1. In order to get a better understanding of eq. (9) we consider the harmonic oscillator. Because this system is one dimensional, its integrability in the Liouville sense is trivial. The conserved quantity is given by the Hamiltonian, $H = \frac{1}{2}(p^2 + \omega^2 q^2)$. By applying eqs. (5) we find the equations of motion $\dot{q} = p$ and $\dot{p} = -\omega^2 q$. These can be described by the Lax pair

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}. \quad (10)$$

Applying eq. (9) we find

$$\dot{L} = \begin{pmatrix} \dot{p} & \omega \dot{q} \\ \omega \dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix} = [M, L]. \quad (11)$$

By comparing the entries of the two matrices in eq. (11) the equivalence to the equations of motion is shown.

The existence of a Lax pair allows for an easy construction of conserved quantities.

Proposition All quantities $O_n \equiv Tr(L^n)$ given by the trace of integer powers of the Lax matrix are conserved,

$$\frac{d}{dt} O_n = 0. \quad (12)$$

Proof. The proposition can be shown easily by explicitly computing the time deriva-

tive \dot{O}_n and using the cyclic property of the trace,

$$\begin{aligned} \frac{d}{dt}O_n &= Tr\left(\frac{d}{dt}L^n\right) = \sum_{i=0}^{n-1} Tr\left(L^i \dot{L} L^{n-i-1}\right) \\ &\stackrel{(9)}{=} \sum_{i=0}^{n-1} Tr\left(L^i [M, L] L^{n-i-1}\right) \\ &= \sum_{i=0}^{n-1} [Tr(L^i M L^{n-1}) - Tr(L^{i+1} M L^{n-i-1})] \stackrel{cyc.}{=} 0. \end{aligned}$$

Example 2. In the case of the harmonic oscillator we can demonstrate how to find the one conserved quantity of the system, the Hamiltonian, by applying eq. (12). By direct calculation we find

$$L^2 = \begin{pmatrix} p^2 + \omega^2 q^2 & 0 \\ 0 & p^2 + \omega^2 q^2 \end{pmatrix}.$$

Therefore the Hamiltonian can be written as $H = \frac{1}{4}Tr(L^2)$. We further remark, that only the trace of some integer powers of the Lax matrix yield interesting results. Looking back at the harmonic oscillator we find that $Tr(L^{2n+1}) = 0$ which is a trivially conserved quantity and $\frac{1}{4}Tr(L^{2n}) = H^n$ which does not provide new conserved quantities.

Since the harmonic oscillator is a one dimensional system and hence only possesses a single conserved quantity, the Poisson commutation stated in Liouville integrability is a trivial consequence. However in a more general case we must further show that the quantities O_n indeed Poisson commute.

A convenient way of writing a matrix of all the Poisson-brackets between the elements of the Lax matrix is via the tensor product,

$$\{L_1, L_2\} \equiv \sum_{ij,kl} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}$$

Proposition The Poisson commutation of the conserved quantities O_n is fulfilled if we can find a matrix r_{12} such that $\{L_1, L_2\}$ takes the special form of

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \quad (13)$$

Where r_{21} is given by the permutation of r_{12} ,

$$r_{12} = \sum_{ij,kl} r_{ij,kl} E_{ij} \otimes E_{kl} \quad \text{and} \quad r_{21} = \sum_{ij,kl} r_{ij,kl} E_{kl} \otimes E_{ij}.$$

We prove the case where given eq. (13) the conserved quantities O_n will indeed Poisson commute.

First we notice the following identity of the Poisson bracket:

$$\{AB, C\} = A\{B, C\} + \{A, C\}B$$

By making use of this identity multiple times, we can rewrite $\{L_1^n, L_2^m\}$ in the following way

$$\begin{aligned} \{L_1^n, L_2^m\} &= L_1^{n-1}L_2^{m-1}\{L_1, L_2\} + L_1^{n-1}L_2^{m-2}\{L_1, L_2\}L_2 + \dots + L_1^{n-1}\{L_1, L_2\}L_2^{m-1} \\ &\quad + L_1^{n-2}L_2^{m-1}\{L_1, L_2\}L_1 + \dots + \{L_1, L_2\}L_1^{n-1}L_2^{m-1} \\ &= \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}\{L_1, L_2\}L_1^pL_2^q \end{aligned}$$

By applying eq. (13) and using $L_1L_2 = L_2L_1$ we can write this in the more compact form

$$\{L_1^n, L_2^m\} = [a_{12}^{n,m}, L_1] - [b_{12}^{n,m}, L_2] \quad (14)$$

with

$$\begin{aligned} a_{12}^{n,m} &= \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}r_{12}L_1^pL_2^q \\ b_{12}^{n,m} &= \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}r_{21}L_1^pL_2^q \end{aligned}$$

By taking the trace of eq. (14) we find that the lefthand side gives us the Poisson-bracket of the conserved quantities O_n and O_m and the righthand, being the trace of a commutator, is zero. Therefore all conserved quantities O_n do indeed Poisson commute.

In this section we have seen that for classical systems which can be written in the special form of a Lax pair, there exists a systematic way of finding conserved quantities by using the Lax matrix as a generating object of the conserved quantities. The conserved quantities are found by taking the trace of integer powers of the Lax matrix. Furthermore the Poisson commutation of the Lax matrix is described by a special r-matrix.

We will find a very similar procedure in the quantum case of the Heisenberg spin chain. Conserved quantities will be generated by the Lax-operator and the commutation relations of the Lax-operator will be described by a special R-matrix.

3 Heisenberg spin chain

This section is adapted from the paper *How Algebraic Bethe Ansatz works for integrable model* by L. D. Faddeev [2].

We recall that the Hilbert space of the isotropic one-dimensional Heisenberg spin chain is given by the tensor product of the individual quantum spin spaces

$$\mathcal{H}_N = \mathfrak{h}_1 \otimes \dots \otimes \mathfrak{h}_N.$$

The Hamiltonian of this particular system is given by

$$H = \sum_n \left(\mathbf{S}_n \cdot \mathbf{S}_{n+1} - \frac{1}{4} \mathbb{I} \right),$$

where we assume the periodic boundary condition $\mathbf{S}_{N+1} = \mathbf{S}_1$.

Our aim in this section is to find a family of N commuting operators and to show that H itself is a part of this family. The computational steps will be closely related to the steps performed in section 2.2

3.1 Conservation of total spin

If we choose to write the spins of the individual electrons in some orthogonal basis along an axis, it is not difficult to see that our system is invariant with respect to rotations around that given axis. This already suggests that the total spin of the system is conserved,

$$S^\alpha = \sum_n S_n^\alpha, \quad [H, S^\alpha] = 0.$$

Proof. First we recall the commutation relations for spin operators and the fact that operators which do not act on a common site of the Hilbert space commute.

$$[S_n^\alpha, S_n^\beta] = i\hbar \epsilon_{\alpha\beta\gamma} S_n^\gamma, \quad [S_n^\alpha, S_m^\beta] = 0 \quad \text{for } n \neq m.$$

By making use of these commutation relations and Einstein summation convention we can compute the commutator between H and S^α ,

$$[H, S^\alpha] = \sum_{n,m} [S_n^j S_{n+1}^j, S_m^\alpha] = i\hbar \sum_n \underbrace{(\epsilon_{j\alpha\gamma} S_n^\gamma S_{n+1}^j + \epsilon_{j\alpha\gamma} S_n^j S_{n+1}^\gamma)}_{=0} = 0.$$

where in the last line we made use of the anti-symmetry of the epsilon tensor.

In order for the system to be integrable we now need to find further $N - 1$ conserved quantities which all commute. In order to achieve this we take a look at the Lax operator.

3.2 Lax operator

The generating object in the case of the isotropic one-dimensional spin chain is given by the Lax operator $L_{n,a}(\lambda)$. The Lax operator is an operator which depends on a complex variable λ called the spectral parameter and acts on states in the tensor product space $\mathfrak{h}_n \otimes V$ where the quantum spin space \mathfrak{h}_n is denoted by the subscript n and the auxiliary space V is denoted by the subscript a . In our case the auxiliary space V is given by \mathbb{C}^2 . Therefore both the quantum spin space as also the auxiliary space are given by the same vector space. It is important to note that, in contrast to the quantum spin space \mathfrak{h}_n , the auxiliary space V has no physical meaning and is only used temporarily in order to compute the conserved quantities. This is very similar to the case of the Lax matrix, where only the entries of the Lax matrix were functions on the phase space. The structure of the matrix corresponded to an auxiliary space and had no physical meaning.

The explicit form of the Lax operator is given by

$$L_{n,a}(\lambda) = \lambda \mathbb{I}_n \otimes \mathbb{I}_a + i \sum_{\alpha \in \{x,y,z\}} S_n^\alpha \otimes \sigma_a^\alpha, \quad (15)$$

where σ_a^α denotes the Pauli matrices acting in the auxiliary space V .

In similarity to the Lax matrix we can also write the Lax operator in a matrix notation

$$L_{n,a}(\lambda) = \begin{pmatrix} \lambda + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda - iS_n^3 \end{pmatrix},$$

where the entries are operators on the quantum spin space \mathfrak{h}_n and the matrix structure corresponds to the auxiliary space V . Alternatively, we can write the Lax operator as

$$L_{n,a}(\lambda) = \left(\lambda - \frac{i}{2}\right) \mathbb{I}_{n,a} + i\mathcal{P}_{n,a}, \quad (16)$$

where \mathcal{P} is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$. This expression makes sense, since \mathfrak{h}_n and V are in fact both given by \mathbb{C}^2 . Explicitly the permutation operator is given in the following form:

$$\mathcal{P} = \frac{1}{2}(\mathbb{I} \otimes \mathbb{I} + \sum_{\alpha} \sigma^{\alpha} \otimes \sigma^{\alpha}), \quad (17)$$

$$\mathcal{P}a \otimes b = b \otimes a \quad \text{for } a \otimes b \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

Eq. (16) will be useful when computing the commutation relation for the Lax operator.

3.3 The RLL relation

In analogy to the classical case of the Lax pair, the commutation relations of the entries of the Lax operator is needed in order to find commuting operators.

Proposition. The fundamental commutation relation (FCR) of the Lax operator is given by

$$R_{a_1, a_2}(\lambda - \mu)L_{n, a_1}(\lambda)L_{n, a_2}(\mu) = L_{n, a_2}(\mu)L_{n, a_1}(\lambda)R_{a_1, a_2}(\lambda - \mu), \quad (18)$$

where the explicit form of the R-matrix is given by

$$R_{a_1, a_2}(\lambda) = \lambda \mathbb{I}_{a_1, a_2} + i\mathcal{P}_{a_1, a_2}.$$

Note that eq. (18) is written in a triple tensor product space $\mathfrak{h}_n \otimes V_1 \otimes V_2$. The two different auxiliary spaces V_1 and V_2 are now denoted by the subscripts a_1 and a_2 .

Furthermore by looking at the explicit expression for the R-matrix and comparing it with eq. (16) we notice that the R-matrix and the Lax operator are related by

$$L_{n, a}(\lambda) = R_{n, a}(\lambda) - \frac{i}{2}\mathbb{I}_{n, a}.$$

Proof. We recall the commutation relation for permutations:

$$\begin{aligned} \mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2} &= \mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1} = \mathcal{P}_{n, a_2}\mathcal{P}_{a_2, a_1} \\ \mathcal{P}_{a_1, a_2} &= \mathcal{P}_{a_2, a_1} \end{aligned} \quad (19)$$

The commutation relations for permutations can be easily shown by explicitly computing the action on a vector $a \otimes b \otimes c$ in the vector space $\mathfrak{h}_n \otimes V_1 \otimes V_2$, e.g. :

$$\begin{aligned} \mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2}a \otimes b \otimes c &= \mathcal{P}_{n, a_1}c \otimes b \otimes a = b \otimes c \otimes a \\ &= \mathcal{P}_{a_1, a_2}b \otimes a \otimes c = \mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1}a \otimes b \otimes c \end{aligned}$$

Making use of this, proving the RLL relation is reduced to commutation relations of permutations.

By using eq. (16) we can rewrite eq. (18), where the lefthand side becomes

$$R_{a_1, a_2}(\lambda - \mu)L_{n, a_1}(\lambda)L_{n, a_2}(\mu) = [(\lambda - \mu)\mathbb{I} + i\mathcal{P}_{a_1, a_2}][(\lambda - \frac{i}{2})\mathbb{I} + i\mathcal{P}_{n, a_1}] \cdot [(\mu - \frac{i}{2})\mathbb{I} + i\mathcal{P}_{n, a_2}].$$

The righthand side can be written analogously. We can now explicitly expand the FCR on both sides. For the left-hand side, this gives

$$\begin{aligned} R_{a_1, a_2}(\lambda - \mu)L_{n, a_1}(\lambda)L_{n, a_2}(\mu) &= (\lambda - \mu)(\lambda - \frac{i}{2})(\mu - \frac{i}{2})\mathbb{I} + i(\lambda - \mu)(\lambda - \frac{i}{2})\mathcal{P}_{n, a_2} \\ &\quad + i(\lambda - \mu)(\mu - \frac{i}{2})\mathcal{P}_{n, a_1} + i(\lambda - \frac{i}{2})(\mu - \frac{i}{2})\mathcal{P}_{a_1, a_2} \\ &\quad + i^2(\lambda - \mu)\mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2} + i^2(\lambda - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2} \\ &\quad + i^2(\mu - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1} + i^3\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2} \end{aligned}$$

The first four terms appear in both expansions of the lhs and the rhs of eq. (18). Concentrating on the last four terms of the expansion we notice that

$$\mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2} = \mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1} \quad \text{and} \quad \mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2} = \mathcal{P}_{n, a_2}\mathcal{P}_{n, a_1}\mathcal{P}_{a_1, a_2}.$$

The zero order term in the spectral parameters of the lhs expansion directly transforms into the zero order spectral parameter term of the rhs expansion. In order to transform the remaining terms of order one in the spectral parameters we first contract both terms containing $\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1}$,

$$\begin{aligned} i^2(\lambda - \mu)\underbrace{\mathcal{P}_{n, a_1}\mathcal{P}_{n, a_2}}_{\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1}} + i^2(\lambda - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2} + i^2(\mu - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1} = \\ i^2(\lambda - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_1} + i^2(\lambda - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2} \end{aligned}$$

We rewrite the last term of the rhs by adding and subtracting μ such that in total we again have three terms of order one in the spectral parameters,

$$i^2(\lambda - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2} = i^2(\lambda - \mu)\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2} + i^2(\mu - \frac{i}{2})\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2}.$$

As a last step we make use of the commutation relation for permutations

$$\mathcal{P}_{a_1, a_2}\mathcal{P}_{n, a_2} = \mathcal{P}_{n, a_2}\mathcal{P}_{n, a_1} = \mathcal{P}_{n, a_1}\mathcal{P}_{a_1, a_2}.$$

Therefore the terms in the lhs expansion of order one in the spectral parameters can be rewritten as

$$i^2(\lambda - \frac{i}{2})\mathcal{P}_{a_1,a_2}\mathcal{P}_{n,a_1} + i^2(\lambda - \mu)\mathcal{P}_{n,a_2}\mathcal{P}_{n,a_1} + i^2(\mu - \frac{i}{2})\mathcal{P}_{n,a_1}\mathcal{P}_{a_1,a_2} ,$$

which correspond to the order one spectral parameter terms of the rhs expansion in eq. (18). Having shown the RLL relation, we can now make use of the Lax operator in order to construct the monodromy matrix $T_{N,a}(\lambda)$ which will act on the whole Hilbert space.

3.4 Monodromy matrix and the RTT relation

The monodromy matrix $T_{N,a}(\lambda)$ is constructed by taking the ordered product of the Lax operator acting on every quantum spin space.

$$T_{N,a}(\lambda) = L_{N,a}(\lambda)L_{N-1,a}(\lambda)\dots L_{1,a}(\lambda) \quad (20)$$

The name monodromy matrix is motivated by the geometric interpretation of the Lax operator as a connection along the spin chain. $L_{n,a}(\lambda)$ defines the transport between sites n and $n + 1$. Therefore by making use of the periodic boundary conditions we can see that the full product in $T_{N,a}(\lambda)$ is a monodromy around our spin chain circle. The monodromy matrix fulfils the same commutation relation as the Lax operator. This commutation relation is also often referred to as the RTT relation,

$$R_{a_1,a_2}(\lambda - \mu)T_{N,a_1}(\lambda)T_{N,a_2}(\mu) = T_{N,a_2}(\mu)T_{N,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu) \quad (21)$$

Proof. In order to show the RTT relation, we recall that two operators which only act non-trivially on different subspaces commute. Therefore we have the following commutation relation:

$$[L_{n,a_1}(\lambda), L_{m,a_2}(\mu)] = 0 \quad \text{for } n \neq m . \quad (22)$$

Making use of eq. (22) it becomes clear that we can rewrite the product $T_{N,a_1}(\lambda)T_{N,a_2}(\mu)$ as

$$\begin{aligned} T_{N,a_1}(\lambda)T_{N,a_2}(\mu) &= L_{N,a_1}(\lambda)L_{N-1,a_1}(\lambda)\dots L_{1,a_1}(\lambda)L_{N,a_2}(\mu)L_{N-1,a_2}(\mu)\dots L_{1,a_2}(\mu) \\ &\stackrel{(22)}{=} L_{N,a_1}(\lambda)L_{N,a_2}(\mu)L_{N-1,a_1}(\lambda)L_{N-1,a_2}(\mu)\dots L_{1,a_1}(\lambda)L_{1,a_2}(\mu) \end{aligned}$$

With $T_{N,a_1}(\lambda)T_{N,a_2}(\mu)$ written in this form, proving eq. (21) is achieved by applying the RLL relation to every pair $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$.

We see that the local commutation relation for the Lax operator can be extended to a global commutation relation for the monodromy matrix.

We now make use of the RTT relation in order to show that the operators which can be extracted from the monodromy matrix commute.

3.5 Extracting commuting operators

In order to remove the non-physical auxiliary space, we take the partial trace of the monodromy matrix $T_{N,a}(\lambda)$ over the auxiliary space V .

Definition. Consider at an operator $A_{ab} = \sum A_{ij,kl} E_{ij} \otimes E'_{kl}$ defined in the tensor space $V_a \otimes V_b$. The partial trace of A_{ab} over the space V_a is then given by

$$Tr_a(A_{ab}) = \sum_{ij,kl} A_{ij,kl} Tr(E_{ij}) E'_{kl}.$$

If we write an operator as the tensor product of two matrices $O = A \otimes B$ where $A \in V_a$ and $B \in V_b$ we find that $Tr_a(O) = Tr(A)B$ is a matrix in V_b , therefore eliminating the space V_a .

We now find that by taking the trace of the monodromy matrix over the auxiliary space V , we end up with a family of operators which purely act on the whole Hilbert space \mathcal{H} ,

$$t(\lambda) = Tr_a(T_{N,a}(\lambda)).$$

These operators indeed commute,

$$[t(\lambda), t(\mu)] = 0.$$

Proof. The commutation of the two families of operators can be shown by using the RTT relation. By inverting the R-matrix we can write

$$T_{N,a_1}(\lambda)T_{N,a_2}(\mu) = R_{a_1,a_2}^{-1}(\lambda - \mu)T_{N,a_2}(\mu)T_{N,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu).$$

Because the two monodromy matrices act on different auxiliary spaces, we can take the trace over V_1 and V_2 separately,

$$\begin{aligned} \text{Tr}_{a_1}(T_{N,a_1}(\lambda)) \text{Tr}_{a_2}(T_{N,a_2}(\mu)) &= \text{Tr}_{a_1,a_2}(R_{a_1,a_2}^{-1}(\lambda - \mu) T_{N,a_2}(\mu) T_{N,a_1}(\lambda) R_{a_1,a_2}(\lambda - \mu)) \\ &\stackrel{\text{cyc.}}{=} \text{Tr}_{a_2}(T_{N,a_2}(\mu)) \text{Tr}_{a_1}(T_{N,a_1}(\lambda)) , \end{aligned}$$

where in the last line we again used the cyclic property of the trace.

For the sake of completeness we must also show the existence of $R_{a_1,a_2}^{-1}(\lambda)$. This can be easily computed. For $\lambda \neq \pm i$ we find

$$R_{a_1,a_2}^{-1}(\lambda) = \frac{1}{\lambda^2 + 1} (\lambda \mathbb{I}_{a_1,a_2} - i \mathcal{P}_{a_1,a_2}).$$

It is important to note that for the points $\lambda = \pm i$ the inverse of $R_{a_1,a_2}(\lambda)$ is not defined. However, since $t(\lambda)t(\mu)$ is a polynomial in λ and μ we can argue that the commutation of $t(\lambda)$ and $t(\mu)$ still holds for $\lambda = \pm i$ via continuity.

In order to be able to extract the commuting operators, we note that the monodromy matrix is a polynomial in λ of degree N . This can be seen by explicitly expanding the individual Lax operators.

$$T_{N,a} = (\lambda \mathbb{I}_N \otimes \mathbb{I}_a + i \sum_{\alpha} S_N^{\alpha} \otimes \sigma_a^{\alpha}) \dots (\lambda \mathbb{I}_1 \otimes \mathbb{I}_a + i \sum_{\alpha} S_1^{\alpha} \otimes \sigma_a^{\alpha})$$

The highest order term λ^N is proportional to the identity. The second highest term λ^{N-1} is proportional to $\sum S_n^{\alpha} \otimes \sigma_a^{\alpha}$. All lower order terms contain product of the terms $(S_i^{\alpha} \otimes \sigma_a^{\alpha})(S_j^{\alpha} \otimes \sigma_a^{\alpha})$.

Therefore by taking the trace of the monodromy matrix over the auxiliary space, we get

$$t(\lambda) = 2\lambda^N + \sum_{n=0}^{N-2} Q_n \lambda^n.$$

The second highest order term vanishes since the Pauli matrices are traceless. We see that the monodromy matrix generates a family of $N - 1$ commuting operators Q_n .

3.6 Conservation of commuting operators

In this section our goal is to show that the $N - 1$ commuting operators Q_n we obtained are indeed conserved. One way of proving this statement is to show that

the Hamiltonian H is part of the Q_n . In order to show that the Hamiltonian H belongs to the family of Q_n , we must take a second look at the Lax operator and determine some special points. By looking at eq. (16) we can see that $\lambda = i/2$ is a special point, since the Lax operator becomes proportional to the permutation operator $L_{n,a}(i/2) = i\mathcal{P}_{n,a}$. Furthermore, the derivative of the Lax operator with respect to λ is equal to the identity, $\frac{d}{d\lambda}L_{n,a}(\lambda) = \mathbb{I}_{n,a}$. Lastly by looking at eq. (17) and remembering that the Pauli matrices are traceless, we see that taking the trace of a permutation operator over the auxiliary space yields the identity $Tr_a(\mathcal{P}_{N,a}) = \mathbb{I}_N$. Therefore we can now compute the monodromy matrix for $\lambda = i/2$

$$T_{n,a}(i/2) = i^N \mathcal{P}_{N,a} \mathcal{P}_{N-1,a} \dots \mathcal{P}_{1,a} \stackrel{(19)}{=} i^N \mathcal{P}_{1,2} \mathcal{P}_{2,3} \dots \mathcal{P}_{N-1,N} \mathcal{P}_{N,a}.$$

In the last line we made use of the commutation relations for permutations in order to rewrite the monodromy matrix in such a way that only the last transposition acts on the auxiliary space, which simplifies taking the trace of the monodromy matrix over V and yields the following expression

$$t_N(i/2) = i^N \mathcal{P}_{1,2} \mathcal{P}_{2,3} \dots \mathcal{P}_{N-1,N}.$$

Making use of the fact that consecutively performing the same transposition twice is equal to the identity, we find the inverse of $t_N(i/2)$ to be given by

$$t_N^{-1}(i/2) = i^{-N} \mathcal{P}_{N,N-1} \mathcal{P}_{N-1,N-2} \dots \mathcal{P}_{2,1}.$$

We are now in a position to show that the Hamiltonian H belongs to the family of Q_n . First, we compute the derivative of the monodromy matrix with respect to λ and evaluate the expression at the special point $\lambda = i/2$.

$$\begin{aligned} \left. \frac{d}{d\lambda} T_{N,a}(\lambda) \right|_{\lambda=i/2} &= i^{N-1} \sum_n \mathcal{P}_{N,a} \dots \mathcal{P}_{n+1,a} \mathcal{P}_{n-1,a} \dots \mathcal{P}_{1,a} \\ &\stackrel{(19)}{=} i^{N-1} \sum_n \mathcal{P}_{1,2} \dots \mathcal{P}_{n-1,n+1} \dots \mathcal{P}_{N-1,N} \mathcal{P}_{N,a} \end{aligned}$$

After taking the trace of the resulting expression over the auxiliary space V , we can multiply with $t_N^{-1}(i/2)$ in order to simplify the result,

$$\left[\frac{d}{d\lambda} t_N(\lambda) \right] t_N^{-1}(\lambda) \Big|_{\lambda=i/2} = \frac{1}{i} \sum_n \mathcal{P}_{n,n+1}$$

By recalling eq. (17) we see that we can write

$$H = \frac{1}{2} \sum_n \mathcal{P}_{n,n+1} - \frac{N}{2}.$$

Therefore the Hamiltonian is given by

$$H = \frac{i}{2} \frac{d}{d\lambda} \ln(t_N(\lambda)) \Big|_{\lambda=i/2} - \frac{N}{2}. \quad (23)$$

This shows that all Q_n do not only commute, but are also conserved operators. By adding e.g. the z-component S^z of the total spin we can enlarge the family to N commuting operators which coincides with the degrees of freedom of the system. This shows that the Heisenberg spin chain is indeed an integrable system.

4 Conclusion

This report mainly explained the basics of the Heisenberg spin chain. The methods of the Lax operator, the monodromy matrix and the auxiliary space used to construct the conserved operators, are very important tools. The same holds for the mathematical methods used when calculating the RLL and RTT relations. Therefore the focus of this report was set on familiarising the objects and methods used, laying the fundamentals in order to understand the other two topics presented, namely “Coordinate Bethe Ansatz for Heisenberg Spin Chain” and especially “Algebraic Bethe Ansatz for Heisenberg Spin Chain”, for which the understanding of the Lax operator and the monodromy matrix are a prerequisite. Therefore I would like to refer to the reports of the two mentioned topics for further reading.

References

- [1] H. Bethe *Zur Theorie der Metalle*. Zeitschrift für Physik 71, 1931
- [2] L.D. Faddeev *How Algebraic Bethe Ansatz works for integrable model*. arXiv:hep-th/9605187
- [3] Lenhard L. Ng *Heisenberg Model, Bethe Ansatz and Random Walks*. Senior Honors Thesis, Harvard University, 1996

- [4] O.Babelon, D. Bernard, M. Talon *Introduction to Classical Integrable Systems*.
Cambridge monographs on mathematical physics