# Classification of Compact Simple Lie Algebras

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May 12, 2018

**Abstract**

In this report, the compact Lie algebras are classified via the classification of complex simple Lie algebras. The classification of semisimple Lie algebras using the knowledge about simple Lie algebras follows. An application of this classification is given using representation theory of simple Lie algebras. Finally, a way to build infinite dimensional Lie algebras is shown in the example of the Kac-Moody algebras. The sections about simple and semisimple Lie algebras are largely based on J.E. Humphreys,[1]. Proofs can be generally found in the said reference. W.Fulton/J.Harris,[2] is mainly used as a second reference. The part covering Kac-Moody algebras can be found in the paper P.Goddard/D.Olive[3]

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I. The classical Lie algebras

i. Complexification of compact simple Lie algebras

In order to classify the compact simple Lie algebras, it might be interesting to note that a real compact simple Lie algebra $L$ can always be complexified via $L \otimes_R \mathbb{C} \cong L'$ to a complex simple Lie algebra, making the underlying field the complex numbers $\mathbb{C}$. This is of particular interest since $\mathbb{C}$ is algebraically closed and of characteristic 0, which makes it easier to work with the concerned complex Lie algebra $L$.

Indeed, during the rest of this report, the underlying field $\mathbb{F}$ of the Lie algebra will always assumed to be of characteristic 0 and algebraically closed.

A good example is: $\text{su}(2) \otimes_R \mathbb{C} \cong \text{sl}(2, \mathbb{C})$

The later introduced notion of Cartan subalgebra can be associated with a subalgebra of the real compact simple Lie algebra by saying, that we call $H_0$ a Cartan subalgebra of the real $L$ if $H=H_0 \otimes \mathbb{C}$ is a Cartan subalgebra of the complex Lie algebra $L'$.

ii. The Classical Lie algebras

The classical Lie algebras constitute the four basic Lie algebra classes that will be seen later on, when classifying the complex simple Lie algebras.

The classical Lie algebras are subalgebras of the general linear algebra $\mathfrak{gl}(V)$, which if $\dim V=\mathbb{N}$, is made up of all the possible $N \times N$ matrices if we interpret $\mathfrak{gl}(V)$ as a vector space.

They are simple Lie algebras and the following way of defining them is up to Lie algebra isomorphism as will be shown later on.

ii.1 $A_\ell$

The first classical Lie algebra $A_\ell$ is represented by the special linear algebra $\mathfrak{sl}(\ell+1, \mathbb{F})$. This is the Lie algebra associated to the Lie group $\text{SL}(\ell+1, \mathbb{F})$. Since $\text{SL}(\ell+1, \mathbb{F})$ is defined as the $\ell + 1 \times \ell + 1$ matrices $X$ for which $\det(X)=1$. This condition translates into the condition\[ x \in \mathfrak{sl}(\ell+1) \iff tr(x) = 0 \]

Thus one can easily construct a basis of $\mathfrak{sl}(\ell+1, \mathbb{F})$ and obtains: $E_{ij}$, for $j \neq i$, where $E_{ij}$ is matrix with a 1 in the $i$-th row and $j$-th column, and $H_i=E_{ii}-E_{jj}$ for $i \leq \ell$ as a basis for $\mathfrak{sl}(\ell+1)$.

The dimension is thus: $\dim \mathfrak{sl}(\ell+1, \mathbb{F})=(\ell + 1)^2 - \ell$

---

\[\text{By taking the derivative, see appendix Lie algebras}\]
ii.2 $B_\ell$

The second classical Lie algebra $B_\ell$ is given by the **orthogonal algebra** $\mathfrak{o}(2\ell+1,\mathbb{F})$. It is important to notice that in this case the elements are endomorphisms on a vector space with an odd dimension. From the condition of $O(\ell+1,\mathbb{F})$, $\forall X \in O(\ell+1,\mathbb{F}) : X^T X = 1$, one obtains the condition for $\mathfrak{o}(\ell+1,\mathbb{F})$:

$$x = -x^T$$

The matrices in the orthogonal algebra are thus antisymmetric and it follows that the trace is 0 as well. One can thus show that $\mathfrak{o}(\ell+1,\mathbb{F}) = \mathfrak{so}(\ell+1,\mathbb{F})$. A basis is:

$$E_{ij} - E_{ji} \text{ for } (1 \leq i < j \leq \ell)$$

The dimension is thus: $\dim \mathfrak{o}(\ell+1,\mathbb{F}) = 2\ell^2 + \ell$

ii.3 $C_\ell$

The classical Lie algebra $C_\ell$ is represented by the **symplectic algebra** $\mathfrak{sp}(2\ell,\mathbb{F})$. Since the symplectic group is defined by all the $2\ell \times 2\ell$ matrices $X$ that satisfy: $X^T S X = S$, where $S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$.

For the Lie algebra one obtains the condition:

$$\forall x \in \mathfrak{o}(2\ell,\mathbb{F}) : sx = -x^T s$$

The basis can be chosen as:

$E_{i,j} - E_{j,i} \text{ for } (1 \leq i \leq \ell)$
$E_{i,j} \text{ for } (1 \leq i \neq j \leq \ell)$
$E_{i,i} \text{ and } E_{i,j} + E_{j,i} \text{ for } (1 \leq i, j \leq \ell)$

The dimension is: $\dim \mathfrak{o}(2\ell,\mathbb{F}) = 2\ell^2 + \ell$

ii.4 $D_\ell$

The last classical Lie algebra is given by $\mathfrak{o}(2\ell,\mathbb{F})$. Note that here the dimension is even. The conditions for a matrix to be in $\mathfrak{o}(2\ell,\mathbb{F})$ is the same as in the case of $B_\ell$.

The dimension is: $\dim \mathfrak{o}(2\ell,\mathbb{F}) = 2\ell^2 - \ell$

II. **Simple and semisimple Lie algebras**

i. Definitions and Properties

A **simple** Lie algebra $L$ is defined as a Lie algebra that has no other ideals than 0 and $L$. The additional condition $[L,L] \neq 0$ has to hold as well. This is demanded in order to exclude 1-dim. Lie algebras and abelian Lie algebras from being simple.

If one defines the **adjoint representation** by:

$$\text{ad}: L \rightarrow \mathfrak{gl}(L), x \mapsto [x, \cdot]$$

which really is a Lie algebra representation (the compatibility with the commutator follows from the Jacobi identity). Then one can easily see, that the adjoint representation (often just called ad-representation) is not 0 and is irreducible (since there are no other ideals except for 0 and $L$).
The ad-representation plays a significant role in the analysis of the structure of a Lie algebra, since it allows to better understand the main properties of representations of complex Lie algebras and to classify complex Lie algebras them, points that will be shown later on and are the main topics of this report.

Using the Killing form $\kappa$, which is defined using the ad-representation, one can now try to define semisimple Lie algebras. One then says, that a Lie algebra is *semisimple*, if and only if the Killing form $\kappa$ is nondegenerate on the given Lie algebra. Form the not here mentioned definition for semisimple Lie algebras, one can directly conclude that simple Lie algebras are semisimple. This means that the Killing form $\kappa$ is nondegenerate on a simple Lie algebra.

If one analyses semisimple Lie algebras over $\mathbb{F}$, algebraically closed, further, one can find a very useful property:

**Theorem**: Let $L$ be semisimple. Then there exist ideals $L_1,\ldots,L_t$, which are simple, such that $L=L_1\oplus\ldots\oplus L_t$. Every simple ideal of $L$ coincides with one of the $L_i$ and the Killing form $\kappa_i$ of $L_i$ is the restriction of $\kappa$ to $L_i \times L_i$.

The fact that a semisimple Lie algebra can be decomposed in a similar way allows it to later classify semisimple Lie algebras via the classification of the simple Lie algebras. The ad-representation of a semisimple Lie algebras is completely reducible on semisimple Lie algebra; a fact that follows directly since the simple Lie algebras in this decomposition are ideals. Weyl's theorem for semisimple Lie algebras goes even a step further and says:

**Theorem**: Let $\phi$ a Lie algebra representation of a semisimple Lie algebra, then: $\phi$ is completely reducible

This theorem together with Schur’s lemma:

**Theorem**: Let $\phi$ be an irreducible Lie algebra representation on $V$, then the only endomorphisms of $V$ commuting with all the $\phi(x)$ for $x \in L$.

is highly useful when analysing representations.

ii. The Chevalley-Jordan decomposition

Let's have a look on the general theory of endomorphism of a vector space $V$. Define a *semisimple endomorphism* as follows:

Say $x \in \text{End}V$ is semisimple if the roots of its minimal polynomial over $\mathbb{F}$ are all distinct. This means in the case of $\mathbb{F}$ being algebraically closed that $x$ is semisimple if and only if it can be diagonalized.

We may say a nonzero endomorphism is *nilpotent* if and only if $\exists n \in \mathbb{N} : x^n = 0, x^{n-1} \neq 0$

The Jordan-Chevalley decomposition now states:

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2It is important to state that the following definition only defines semisimple Lie algebras if $\mathbb{F}$ is algebraically closed. The more general definition using the notion of solvability can be found in J.E.Humphreys, Introduction to Lie Algebras and Representation Theory

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If \( x \in \text{End} V \), then \( \exists x_s, x_n \in \text{End} V \) that are unique, \( x_s \) semisimple, \( x_n \) nilpotent, such that: \( x = x_s + x_n \). The two parts of the decomposition commute.

If one now applies this decomposition onto the ad-representation, one obtains ad-semisimple and ad-nilpotent endomorphisms on \( L \). By referring to this decomposition, one can then say, that the element which generates this ad-semisimple endomorphism is called semisimple as well, just as one can call the element of \( L \) corresponding to the ad-nilpotent endomorphism on \( L \), nilpotent. This is now called the \textbf{abstract} Chevalley-Jordan decomposition. By taking a look at the classical Lie algebras from before, one might wonder if this new abstract decomposition corresponds to the direct decomposition if we see this Lie algebra as endomorphisms on \( V \). The following theorem assures that this is indeed the case:

**Theorem:** Let \( L \subseteq \text{gl}(V) \) be a semisimple linear Lie algebra ( \( V \) finite dimensional). Then \( L \) contains the semisimple and nilpotent parts in \( \text{gl}(V) \) of all its elements and the abstract and direct Chevalley-Jordan decompositions in \( L \) coincide.

One can even go a step further and conclude:

**Corollary:** If \( s \) and \( n \) are the semisimple and nilpotent elements of the abstract Chevalley-Jordan decomposition of \( x \), then for any representation \( \phi \) one obtains the direct Chevalley-Jordan decomposition of \( \phi(x) = \phi(s) + \phi(n) \).

Since a representation \( \phi \) is compatible with the commutator one can say that \( \forall x, y \in L: [\phi(x), \phi(y)] = 0 \iff [x, y] = 0 \) thus two commuting elements of a Lie algebra yield, via the representation \( \phi \), two commuting endomorphisms.

### iii. Example: sl(3, \( \mathbb{F} \))

In order to illustrate some of the introduced definitions and theorems, one might want to take a look at the simple Lie algebra \( \text{sl}(3, \mathbb{F}) \). By taking the standard basis defined as in I.i.1 which is here given by:

\[
\begin{align*}
  x_1 &= E_{1,2}, & x_2 &= E_{3,2}, & x_3 &= E_{1,3} \\
  y_1 &= E_{2,1}, & y_2 &= E_{3,2}, & y_3 &= E_{3,1} \\
  h_1 &= E_{1,1} - E_{2,2}, & h_2 &= E_{2,2} - E_{3,3}
\end{align*}
\]

The elements \( h_1 \) and \( h_2 \) are semisimple and thus \( \text{ad} h_1 \) and \( \text{ad} h_2 \) are semisimple, too. The remaining elements are nilpotent and so are the associated elements of the ad-representation. The endomorphisms on \( L \) obtained by the ad-representation for \( h_1 \) and \( h_2 \) yield the following:

\[
\begin{array}{cccccccc}
\{,\} & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & h_1 & h_2 \\
\hline
h_1 & 2 & 1 & -1 & -2 & -1 & 1 & 0 & 0 \\
\hline
h_2 & -1 & 2 & 1 & 1 & -1 & -1 & 0 & 0
\end{array}
\]

It is clear that they are semisimple. But now the question arises if the semisimple elements of a simple Lie algebra always commute or if this is only the case in this precise example. Furthermore,

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3For the full understanding of why this happens, it is important to say that \( \text{ad} L = \text{Der} L \), since only this guarantees that the \( \text{ad} L \) contains the semisimple and nilpotent parts of all its elements. This and the associated definitions are not treated and can be found in the main reference.
one can calculate that \([x_i, y_i] = h_i\) which is another intriguing fact. This leads in a natural way to the analysis of similar subalgebras as the one spanned by \(h_1\) and \(h_2\). Additionally, one would like to somehow split the vector space of the Lie algebra up in a way that is compatible with this result. This idea is the same as when one splits the vector space \(V\) up into eigenspaces \(V_\lambda\) according to the eigenvalues \(\lambda\) of a given endomorphism \(\psi\) on \(V\) which is diagonalizable. This will lead to the introduction of roots and weights.

### III. Roots and Root Systems

#### i. Toral subalgebra

As seen in the example, define a subalgebra of \(L\) by the span of the semisimple elements of \(L\). This is obviously not 0 since the \(L\) is semisimple. If there were no semisimple elements in \(L\) this would mean that \(L\) is only made up of nilpotent elements, thus the adjoint representation of all elements in \(L\) would be nilpotent as well. This is not possible, since Engel’s theorem\(^4\) states that then the algebra is nilpotent and thus solvable. This is a contradiction with the definition of the semisimple Lie algebra. Call the subalgebra obtained in this way a **toral subalgebra** \(T\).

Call a **maximal toral subalgebra** \(H\) a toral subalgebra that is not properly included in any other. Now there exists a rather handy lemma which clearly defines any toral subalgebra as being abelian.

**Proof:** Assume \(x, y \in T\) and nonzero. As seen before, we may say that any two elements in \(T\) commute if the ad-representation restricted to \(T\) \(\text{ad}_T x\) is 0, this means since it can be diagonalized that the eigenvalues are all 0. Now assume: \([x, y] = ay\) with a nonzero. Thus \([y, x] = -ay\) and this leads to: \([y, [y, x]] = 0\). Since \(\text{ad}_T y\) is diagonal as well and nonzero, the eigenvectors span \(T\) and the eigenvalues are not all zero. So if one writes the \(x\) as linear combination of eigenvectors of \(\text{ad}_T y\), one can calculate that \([y, [y, x]]\) is nonzero according to the last assumptions. But this is a contradiction to the previous result and thus the lemma holds. \(\Box\)

Note this definition for a toral subalgebra is given for \(\mathbb{F}\) algebraically closed. For a given \(\mathbb{F}\) a toral subalgebra is composed of commuting semisimple elements. In other words, in general a toral subalgebras is made up of commuting semisimple elements and in this specific case the elements commute as a result of being semisimple and the fact that \(\mathbb{F}\) is as we have chosen it to be.

Since \(H\) is abelian, any two endomorphisms obtained from a representation \(\phi\) commute. This applied to the ad-representations now shows, that \(\text{ad}_H H\) is made up of commuting and diagonalizable endomorphisms. This leads to the conclusion that \(\text{ad}_L H\) is simultaneously diagonalizable; in other words, we can find a basis of \(L\) such that for any \(h \in H\) \(\text{ad}_L h\) is diagonal. Thus one can write \(L\) as the direct sum of eigenspaces of a given \(\text{ad}_L h\) and for any other \(h'\) this turns out to be the eigenspace decomposition, too. Since it is not possible to denote the eigenspaces by the corresponding eigenvalue of \(\text{ad}_L h\) without given more and unjustified attention to this certain \(h\), one defines elements \(\alpha \in H^*\) for which \(\alpha(h)\) is exactly the eigenvalue for a certain eigenspace. Thus the decomposition of \(L\) as given by \(\text{ad}_L\) is diagonal.

Define \(\alpha \in H^*\) which are nonzero as **roots** and the set of roots is denoted by \(\Phi\). The decomposition one obtains then is called the **Cartan decomposition** or **root space decomposition**:
\[
L = C_L(H) \bigoplus \bigoplus_{\alpha \in \Phi} L_\alpha.
\]
One can go on and show that the Killing form \(\kappa\) is nondegenerate on the centralizer of \(H\). Using this fact along with other arguments, one can then show that \(H = C_L(H)\)

\(^4\)see [1]
and thus the Killing form \( \kappa \) is nondegenerate on \( H \).
So one can say that the maximal toral subalgebra \( H \) of a semisimple Lie algebra \( L \) yields a unique root space decomposition via the \( \text{ad} \)-representation and defines elements in its dual space that are called roots. In the coming chapters the importance of \( H \) in order to classify semisimple Lie algebras will become clearer.

ii. Roots

The roots \( \alpha \in H \) obtained in the previous section exhibit a clear structure that will become even more visible when constructing a root system. Indeed, the roots together with the Killing form \( \kappa \) yield many interesting results that are analysed in this section. The importance of Killing form \( \kappa \) can be partly seen in the role it plays connecting roots \( \alpha \in H^* \) to elements \( h \in H \) via:

Say \( \phi \in H^* \) is connected to \( t_\phi \in H \) by: \( \phi(h) = \kappa(t_\phi, h) \). This is a 1-1 one correspondence between \( H \) and \( H^* \) since the Killing form \( \kappa \) is nondegenerate on \( H \) and is bilinear.

Some of the first observations might be that for any two roots \( \alpha, \beta \in H^* \) with \( \alpha + \beta \in \Phi \), \([L_\alpha, L_\beta] \subset L_{\alpha + \beta} \). This follows directly from the Jacobi identity:

\[
\text{for } h \in H \text{ ad } h([x, y]) = [h, [x, y]] = -[x, [y, h]] - [y, [h, x]] = [x, [h, y]] + [[h, x], y] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y] 
\]

One finds, since there are only a finite number of roots (since we assume the Lie algebra finite-dimensional) any \( \text{ad} \) \( x \) for \( x \in L_\alpha \) is nilpotent.
Using the fact that \( \kappa([x, y], z) = 0 \), one obtains that for two roots \( \alpha, \beta \) such that \( \alpha + \beta \neq 0 \): \( L_{\alpha} \) and \( L_{\beta} \) are orthogonal relative to \( \kappa \). It follows, that for \( \alpha \in H^* \): \( \kappa(H, L_{\alpha}) = 0 \).

There are even many more properties that can be found for roots. Some of them are the following:
(a) \( \Phi \) spans \( H^* \)
(b) Let \( \alpha \in \Phi \), then \( -\alpha \in \Phi \)
(c) Let \( \alpha \in \Phi \), \( x \in L_\alpha, y \in L_{-\alpha} \), then: \( [x, y] = \kappa(x, y)t_\alpha \)
(d) If \( \alpha \in \Phi \), then \([L_\alpha, L_{-\alpha}] \) is one dimensional according to (c) and spanned by \( t_\alpha \)
(e) \( \alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = 0 \)
(f) If \( \alpha \in \Phi \) and \( x_\alpha \in L_\alpha \) nonzero, then there is a unique \( y_\alpha \in L_{-\alpha} \) such that \( x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha] \) span a three dimensional simple subalgebra \( S_\alpha \) isomorphic to \( sl(2, \mathbb{F}) \) via: \( x_\alpha \mapsto E_{1,2}, y_\alpha \mapsto E_{2,1}, h_\alpha \mapsto H_{1,1} \) with the notation used in the description of \( A_\ell \)
(g) \( h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \); \( h_\alpha = -h_{-\alpha} \)

It is clear that from the definition of \( h_\alpha \) follows: \( \alpha(h_\alpha) = 2 \). From those facts, the statement about the isomorphism between \( S_\alpha \) and \( sl(2, \mathbb{F}) \) leads to additional and very important properties of roots.

Indeed, take the representation of \( S_\alpha \), where \( S_\alpha \) is defined as above, on the space \( M \) spanned by \( H \), the maximal toral subalgebra and the spaces \( L_{ca} \), where \( c \) is a nonzero scalar. If one acts with \( \text{ad} h_\alpha \) on an element of \( M \) without \( H \), one obtains as weight of the representation \( ca(h_\alpha) = 2c \), since we have seen that \( \alpha(h_\alpha) = 2 \). But this has to satisfy the representation theory of \( sl(2, \mathbb{F}) \), studied in the appropriate appendix, since \( S_\alpha \) is isomorphic to this simple Lie algebra. Thus the weights have to be integers which implies that \( c \) is a multiple of \( 1/2 \). Furthermore, the representation can be decomposed into irreducible ones, according to Weyl’s theorem. We exactly now how
those irreducible representations behave. The amount of irreducible representations is given by the dimension of the weight space to the weight 0 and the dimension of the weight space to the weight 1. For the dimension of the weight space 0, we obtain that it is given obviously by the dimension of H minus one to account for the span of $h_k$ in $S_0$. Since the ad $h_k$ is zero on this subalgebra, plus 1 obtained by the ad-representation on $S_0$. If now there was a root $\beta = c\alpha$, $\beta(h_k) = c\alpha(h_k) = \kappa(t_\beta, h_k) = c\kappa(t_\alpha, h_k) \rightarrow t_\beta = ct_\alpha$ which leads to $[L_\beta, L_{-\beta}] = [L_\alpha, L_{-\alpha}]$. Thus for a given c and its negative, the associated root spaces cannot generate a Lie algebra that is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ and be in this M via a direct sum, since we know that $S_0$ is already a member of this decomposition into a direct sum. Thus the $S_{ca}$ are not in the decomposition. This is the case since the ad-representation of $S_0$ on $S_0$ is irreducible. So there are no other $L_\alpha$ since this would mean that Weyl’s theorem does not hold with is a contradiction. We obtain: if $H_\alpha$ is H without $H_\alpha, M = H_\alpha \oplus S_0$. Thus we know that the only multiples of \( \alpha \) are \( \alpha \) and $-\alpha$. The $L_\alpha$ are one dimensional according to the representation of $\mathfrak{sl}(2, \mathbb{F})$.

If we try to apply the ad-representation of $S_0$ onto the $L_\beta$ where $\beta$ is not a multiple of $\alpha$, we obtain by acting on $K = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$, that these $L_{\beta + i\alpha}$ are one dimensional as seen before and that $\beta + i\alpha \neq 0$. So K is completely reducible and the weights are $\beta(h_k) + 2i$, if of course $\beta + i\alpha \in \Phi$. As one can see, either the weight 0 or the weight 1 can show up but not both and every weight occurs only once, thus the representation is irreducible on K and we can say that $\alpha$-string through $\beta$ is unbroken. This means that if $\beta + qa$ and $\beta - ra$ are extremal roots, for $q, r \in \mathbb{Z}^+$, then $\beta + ia$ is a root for $-r \leq i \leq q$. Because of the symmetric occurrence of weights around 0 we obtain: $(\beta - ra)(h_k) = -(\beta + qa)(h_k)$ which yields: $\beta(h_k) = r - q$.

In this case ad $x_\alpha(x_\beta) = x_{\alpha + \beta}$ for $\alpha + \beta \in \Phi$, just as in the $\mathfrak{sl}(2, \mathbb{F})$ case. Thus: $[L_\alpha, L_\beta] = L_{\alpha + \beta}$. To sum up:

**Proposition:**
(a) $\alpha \in \Phi$ implies dim $L_\alpha = 1$. In particular, $S_0 = L_\alpha + L_{-\alpha} + H_\alpha$, where $H_\alpha = [L_\alpha, L_{-\alpha}]$ and for given nonzero $x_\alpha \in L_\alpha$, there is a unique $y_\alpha \in L_{-\alpha}$ satisfying $[x_\alpha, y_\alpha] = h_\alpha$
(b) If $\alpha \in \Phi$, the only scalar multiples of $\alpha$ in $\Phi$ are: $\pm \alpha$.
(c) If $\alpha, \beta, \alpha + \beta \in \Phi$, then $\beta(h_k) \in \mathbb{Z}$ and $\beta - \beta(h_k)\alpha \in \Phi$.
(d) If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha + \beta}$.
(e) Let $\alpha, \beta \in \Phi, \beta \neq \pm \alpha$. Let $r$ and $q$ be the respective largest integers for which $\beta - ra$ and $\beta + qa$ are roots. Then all $\beta + ia \in \Phi(-r \leq i \leq q)$, and $\beta(h_k) = r - q$.
(f) L is generated as a Lie algebra by the root spaces $L_\alpha$.

Taking a look at $\beta(h_k)$ one can notice: $\beta(h_k) = \kappa(t_\beta, h_k) = \kappa(t_\beta, \frac{2t_\alpha}{\kappa(t_\alpha, t_\beta)}) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\beta)} \in \mathbb{Z}$.

The Killing form $\kappa$ is a bilinear symmetric nondegenerate form on L and we can use it to construct a bilinear symmetric nondegenerate form $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$. Thus $\beta(h_k) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Now we have seen many properties of roots but it is still not easy to see a very clear structure. It would be nice if the roots not only spanned $H^*$, but would span an euclidean space which has a well visible structure. That this is the case is shown in the next paragraph.

First choose a basis of $H^*$ using roots $\alpha_1, ..., \alpha_\ell$ where $\ell$ is the dimension of $H^*$. Now we show that any root $\beta$ can be written in this unique way: $\beta = \sum_{i=1}^{\ell} c_i(\alpha_i)$. In order to prove this, we start:

For each $j = 1, ..., \ell$ we have $(\beta, \alpha_j) = \sum_{i=1}^{\ell} c_i(\alpha_i, \alpha_j)$. Multiplying this by $2/(\alpha_j, \alpha_j)$, one obtains:
We have seen that the roots span a real euclidean space $E$ and satisfy the conditions of the theorem. We can define a metric and thus that this $E$ is a vector space. This is easily done using the inner product $\langle \alpha, \beta \rangle = \sum_{a \in \Phi} (a, \alpha)(a, \beta)$, which in particular yields for $\beta \in \Phi$:

$$\langle \beta, \beta \rangle = \sum_{a \in \Phi} (a, \beta)^2.$$ 

This means a semisimple Lie algebra $L$ with maximal toral subalgebra $H$ and roots $\Phi$ spanned by the roots, which is possible, since $\mathbb{Z} \subset \mathbb{Q}$. Then we have:

$$Q: \dim Q = \ell = \dim_F H^*.$$ 

Since for any $\lambda, \mu \in H^*$:

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \sum_{a \in \Phi} \kappa(t_a, t_\lambda)\kappa(t_a, t_\mu) = \sum_{a \in \Phi} (\alpha, \lambda)(\alpha, \mu),$$

which in particular yields for $\beta \in \Phi$:

$$\langle \beta, \beta \rangle = \sum_{a \in \Phi} (a, \beta)^2.$$ 

Thus one obtains:

$$\langle \alpha, \beta \rangle = \sum_{a \in \Phi} (a, \alpha)(a, \beta) = \sum_{a \in \Phi} (a, \alpha)(a, \beta).$$

This shows that the Killing form $\kappa$ induces an inner product on this $E_Q$ which in turn means that we can define a metric and thus that this $E_Q$ is euclidean. It is now easy to extend this to a real euclidean space $E$ via: $E = E_Q \otimes \mathbb{R}$. This means a semisimple Lie algebra $L$ with maximal toral subalgebra $H$ and roots $\Phi$ yield in a natural way a real euclidean space $E$ (via Killing form $\kappa$) with the following properties:

**Theorem:** $L$, $H$, $\Phi$ and $E$ as above. Then:

(a) $\Phi$ spans $E$, 0 is not a root.

(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and this is the only scalar multiple that is a root.

(c) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Phi$.

(d) If $\alpha, \beta \in \Phi$, then $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

### iii. Root Systems

We have seen that the roots span a real euclidean space $E$ and satisfy the conditions of the theorem above. Now we take the axiomatic approach and call $\Phi$ a **root system** that has the properties mentioned before. This root system is what is analysed in this section. The dimension of $E$ is called the **rank** and is equal to the dimension of the maximal toral subalgebra $H$ of a semisimple Lie algebra. The $\ell$ in the index of the classical Lie algebras $A\ell$ etc. indicates the rank of the associated root system. In addition we introduce a new notation for $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \equiv \langle \alpha, \beta \rangle$. This is not a symmetric form anymore and is only linear in the first term.

Since we have said that $E$ is a real euclidean space, it would be nice to represent it as we usually do with euclidean spaces, namely be able to draw the roots as vectors and to be able to define an angle between them. In order to do this, one needs a scalar product which is defined on the given vector space. This is easily done using the inner product $\langle \alpha, \beta \rangle = \| \alpha \| \| \beta \| \cos \theta$, where $\theta$ is the angle between the roots and $\| \alpha \|^2 = \langle \alpha, \alpha \rangle$.

Now we know that $(\beta, \alpha) \in \mathbb{Z}$, which, by doing some small changes, means that: $\langle \beta, \alpha \rangle = 2 \frac{\| \beta \|}{\| \alpha \|} \cos \theta$. Thus one obtains: $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2 \theta \in \mathbb{Z}$. The constraints are: $0 \leq \cos^2 \theta \leq 1$ and $4\cos^2 \theta$ is an integer. This means that $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ have the same sign and that we can sum the solutions up in the table 1.

This is an important result. Indeed, this allows us to draw roots systems for semisimple Lie alge-
bras and to exclude a whole lot of situations that cannot belong to a root system of a semisimple Lie algebra. Furthermore the following lemma is a direct result:

**Lemma**: Let $\alpha, \beta$ be nonporoprtional roots. If $(\alpha, \beta) > 0$ (if the angle between the roots is strictly acute), then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.

Additionally, one obtains again: The $a$-string through $\beta$ is unbroken, from $\beta - r\alpha$ to $\beta + q\alpha$, and $\langle \beta, \alpha \rangle = r - q$ just as before. The new thing we learn is that the length of such a root string is of at most 4.

In order to further reduce the information we need to describe a root system and by these means a semisimple Lie algebra, it occurs to be natural to choose a basis of $E$ composed of roots. But as can be seen afterwards, it is advantageous to not only limit our choice to any roots but to add an additional condition.

Call a **base** $\Delta \subset \Phi$ of the root system if:

(a) $\Delta$ is a basis of $E$
(b) $\forall \beta \in \Phi : \beta = \sum_{\alpha \in \Delta} k_{\alpha} \alpha$ is unique and all the $k_{\alpha}$ are either nonpositive or nonnegative.

---

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$(\beta, \alpha)$</th>
<th>$\theta$</th>
<th>$\frac{|\beta|^2}{|\alpha|^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\pi/2$</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\pi/3$</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$2\pi/3$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\pi/4$</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>$3\pi/4$</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$\pi/6$</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
<td>$5\pi/6$</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 1**: For $\alpha \neq \pm \beta$
The elements of \(\Delta\) are called **simple roots**. The roots are now divided into **positive roots**, given by the set \(\Phi^+\), and **negative roots**, given by the set \(\Phi^-\). We have that: \(\Phi = \Phi^+ \cup \Phi^-\) and \(0 = \Phi^+ \cap \Phi^-\).

This definition is not very clear in the sense of its existence. Indeed, it is clear that a basis made up of roots should exist, since this is just what we have shown before and how the real euclidean space \(E\) has been constructed. Unfortunately, there is nothing guaranteeing that the base can be chosen to satisfy the second condition. It is thus important that this theorem holds:

**Theorem:** A base \(\Delta\) of a root system always exists.

In order not to prove but at least motivate this statement and to show how one can easily construct such a base, it is useful to introduce some additional definitions:

Call the **reflecting hyperplane** \(P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}\) the plane of all the elements orthogonal to \(\alpha\). This allows to define a **reflection** \(\sigma_\alpha\) which acts in the following way: \(\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{\alpha, \alpha}\). This is actually one of the conditions a roots system has to satisfy, so this reflection is nothing else than the expression of this condition in a geometrical way.

Now those hyperplanes \(P_\alpha\) divide the euclidean space \(E\) into different connected components \(E - \bigcup_{\alpha \in \Phi} P_\alpha\), called the open **Weyl chambers** \(C\) of \(E\). We call an element \(\gamma \in E\) **regular**, if \(\gamma\) is not an element of \(\bigcup_{\alpha \in \Phi} P_\alpha\). The of course \(\gamma\) lies in a unique Weyl chamber that can be designated by \(C(\gamma)\).

Since the reflections \(\sigma_\alpha\) are equivalent to one of the conditions of a roots system, it follows that \(\sigma_\alpha \Phi = \Phi\). The group of reflections \(\sigma_\alpha\) is called the **Weyl group** \(W\) of \(E\) and is a subgroup of all the permutations of the elements of the set \(\Phi\). \(W\) leaves the \((\alpha, \beta)\) invariant.

**Lemma:** If \(\Delta\) is a base of \(\Phi\), then \((\alpha, \beta) \leq 0\) and \(\alpha - \beta\) is not a root.

Now using this new notions, one can find the base \(\Delta\) of a root system the following way:

First one takes a regular \(\gamma \in E\) and constructs the hyperplane \(P_\gamma\). Now the roots lying in the same semispace as \(\gamma\) are will be our positive roots \(\Phi^+(\gamma)\) and the other ones the negative roots \(\Phi^-(\gamma)\). In the set \(\Phi^+(\gamma)\) we now have to choose the roots that are simple. Since all the elements in this set have to be positive one can argue that the roots with the smallest angles between them and the hyperplane \(P_\gamma\) (this means they have the biggest angle between them and \(\gamma\)) and that are linear independent are those simple roots. This follows from the fact that we cannot form them by adding up other positive roots (assume we could, where \(\alpha = \beta_1 + \beta_2\), then \((\gamma, \alpha)\) is bigger than \((\gamma, \beta_1)\) and \((\gamma, \beta_2)\) which are linearly independent and the three together only span a subspace of dimension 2) and that no root lies in the hyperplane \(P_\gamma\) since \(\gamma\) is regular. This construction together with the existence theorem guarantees that this is indeed a base \(\Delta(\gamma)\). The Weyl chamber \(C(\gamma)\) is then called a **fundamental Weyl chamber relative to** \(\Delta\). Taking a look on how the reflections \(\sigma_\alpha\) behave, we find that they map one Weyl chamber to another and thus the fundamental Weyl chamber to another Weyl chamber. This shows that when acting on a base \(\Delta\) one obtains a new base. The Weyl group generates all the possible bases of a root systems, which are the image of all possible reflections of a given base.

As can be seen in the examples, there are two types of root systems, **irreducible** and **reducible** root systems. We call a root system **reducible** if we can find two subsets \(\Phi_1, \Phi_2 \subset \Phi\), such that: \(\Phi = \Phi_1 \cup \Phi_2, 0 = \Phi_1 \cap \Phi_2\) and \(\forall \beta \in \Phi_1 \forall \alpha \in \Phi_2 : (\beta, \alpha) = 0\). That means we have two sets that are orthogonal one to another. We say a root system is **irreducible** if it is not reducible. Making a difference at this points seems to be unjustified but we will show that irreducible root systems
Figure 2: The root system $A_2$. The dotted lines represent the hyperplanes. The hatched area is the fundamental Weyl chamber to the given base. Image taken from [1]

behave towards reducible ones in a similar way to how simple Lie algebras behave towards semisimple Lie algebras that are not simple; the former are the building blocks of the latter. Directly in this section we show a number of important properties of irreducible root systems.

First let us define a **partial ordering** $\prec$ on $E$. This is rather easy to do with what we have seen so far. We have said a root $\beta$ is positive if and only if: $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ for $k_\alpha \geq 0$, which we will write $0 \prec \beta$. Now we say: $\alpha \prec \beta$ if and only if $0 \prec \beta - \alpha$. This indicates that in general there is not a global maximal element $\gamma$.

**Lemma**: Let $\Phi$ be irreducible. Relative to the partial ordering $\prec$, there is a unique maximal root $\beta$. In particular if $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, then all $k_\alpha > 0$.

**Lemma**: Let $\Phi$ be irreducible. Then $W$ acts irreducibly on $E$. In particular, the $W$-orbit of a root $\alpha$ spans $E$.

**Lemma**: Let $\Phi$ be irreducible. Then at most two root lengths occur in $\Phi$, and all roots of a given length are conjugate under $W$. We then just talk about the long respectively short root.

This lemma follows directly from the table 1 and the fact and the fact that there are no other squared length rations other than the ones given there. Indeed, if there were more than two a ratio similar to $3/2$ should occur. Of course one might want to say that if we have the roots $\alpha, \beta, \gamma$ that have the ratio $2$ between $\alpha$ and $\beta$ and the ratio $3$ between $\beta$ and $\gamma$ and that $\alpha$ and $\gamma$ are orthogonal one to the other so that we cannot give the ratio as given in the table. But $W$ allows us to take either $\alpha$ or $\gamma$ and to map it to a root $\delta$ of the same length that is not orthogonal to either $\gamma$ or $\alpha$ which then should yield a ratio of $3/2$ which is forbidden.

**Lemma**: Let $\Phi$ be irreducible. In the case that there are two root lengths that occur, the unique maximal roots $\beta$ will be long.

It follows form the fact that $W$ acts irreducible on irreducible root systems, that we can use it to describe a reducible root system. Indeed a given $\Phi$ can be written as: $\Phi = \Phi_1 \cup \ldots \cup \Phi_n$, there the $\Phi_i$ are irreducible and that the corresponding $E$ is: $E = \bigoplus_i E_i$ which originates from splitting $E$.
which is then completely reducible as representation, into irreducible $E_i$. This is a very important fact used later during the classification of semisimple complex Lie algebras.

IV. Classification Theorem and Consequences

In this chapter, we introduce the Cartan matrices and the Dynkin diagrams as tools to classify the semisimple Lie algebras. Then the classification theorem is presented along with some very important theorems. Finally an example, where to some extend the utility of this classification appears, is given.

i. Cartan Matrices and Dynkin Diagrams

We define the Cartan matrix of a given root system by:

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle,$$

where the $\alpha_i, \alpha_j$ are simple roots. The Cartan matrix is thus a $\ell \times \ell$ matrix. The entries $\langle \alpha_i, \alpha_j \rangle$ are called Cartan integers. Indeed, the Cartan matrix has only 2's on the diagonal and negative integers form the table 1 on the minor diagonal. The other entries are zero as will become clear after the classification theorem.

Additionally, we see the Cartan matrix is not unique but given up to this artificial numbering of the simple roots. In turn we know everything we need about a given root system once we know the simple roots. Indeed, this is why we did not choose a given basis of roots but added this additional condition where we wanted to be able to split the roots into negative and positive roots. Of course, the Cartan matrix does not inform us about the length of a given simple root etc. but just about the ratios. It turns out that this defines thus the root system up to isomorphism. In other words, the Cartan matrix is a way to write a root systems up to isomorphism in a more compact way. This is given by:

**Proposition:** Let $\Phi \subset E'$ be another root system with base $\Delta' = \{a_1',...,a_\ell'\}$. If $\langle a_i', a_j' \rangle = \langle a_i, a_j \rangle$ for $1 \leq i, j \leq \ell$, then the bijection $a_i \mapsto a_i'$ extends to an isomorphism $\phi : E \rightarrow E'$ mapping $\Phi$ onto $\Phi'$ and satisfying: $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for all $\beta, \alpha \in \Phi$. Therefore, the Cartn matrix of $\Phi$ determines $\Phi$ up to such an isomorphism.

In the case of a reducible root system, the Cartan matrix is made up of Cartan matrix blocks of irreducible root systems on the diagonal. This is a direct consequence of the different irreducible subsets being orthogonal one to another. Thus the Cartan integers of simple roots of different irreducible root systems are 0.

The Dynkin diagram of a root system can be determined using the Cartan matrix and is very intuitive once one knows how it is composed. A Dynkin diagram is composed of $\ell$ vertices which correspond to the rank of the root system and the i-th and j-th vertices are connected by $\langle a_j, a_i \rangle$ lines ($a_i, a_j$ are simple roots).

Since this does not determine which root is longer in the case of two different lengths, one adds an arrow pointing in the direction of the longer roots.

Here again irreducible root systems yield a connected strings and reducible root systems a set of different non-interconnected strings which belong to a given irreducible root system. This is a direct consequence of the form the Cartan matrices take in that case. Then again all this goes back to semisimple Lie algebras being made up of simple ones.
ii. Classification theorem

Since we have seen how a reducible root system is built using irreducible ones, it is sufficient to classify the irreducible root systems. This gives:

**Classification theorem:** If $\Phi$ is an irreducible root system of rank $\ell$, its Dynkin diagram is one of the following (having $\ell$ vertices):

![Figure 3: All possible Dynkin diagrams. Usually ones says that for A$_\ell$: 1 $\leq$ $\ell$; B$_\ell$: 2 $\leq$ $\ell$; C$_\ell$: 3 $\leq$ $\ell$; D$_\ell$: 4 $\leq$ $\ell$. Of course here: $\ell$ is given by n. Image taken from Wikipedia.](image)

There are now a few remarks. First it is impressive how only a few possibilities remain and that four of them correspond to the four classical Lie algebras. These are the only ones that are given for any rank $\ell$. It is important to note that of course the Dynkin diagrams which are given for an $\ell$ bigger than a given number do already exist for $\ell$ smaller but in that case share the same Dynkin diagram with another classical Lie algebra of same rank. This is well visible for $\ell = 1$. What this means can be seen later.

Then there are some Dynkin diagrams that only occur for a given rank $\ell$. Note that they are found simply because they can exist, there is nothing that prohibits the existence of such a Dynkin diagram. The question is now of course: Do they belong to irreducible roots systems? The answer is given by:

**Theorem:** For each Dynkin diagram (or associated Cartan matrix) of typ A-G, there exists an irreducible root system having the given diagram

So this means that we have a connection between the Dynkin diagrams, the Cartan matrices and the roots systems. Indeed, as mentioned often enough the irreducible root systems and the respective Cartan matrices and Dynkin diagrams are the components of the general cases. Now we have yet to specify the connection of simple and thus semisimple Lie algebras to those irreducible respectively reducible Lie algebras, in order to completely achieve a classification. The connection from simple Lie algebra to irreducible root system is given by:

**Theorem:** Let $L$ be a simple Lie algebra, $H$ a maximal toral subalgebra and $\Phi$ the set of roots. Then $\Phi$ is an irreducible root system as we have defined it.

The link between simple and semisimple Lie algebras is already known, so we have that any semisimple Lie algebra has a (reducible, if not simple) root systems.
As we have seen before, the Cartan matrices and thus the Dynkin diagrams do not yield a unique
root system but determine it up to isomorphism. Now consider:

**Theorem:** Let \( L, L' \) be two simple Lie algebras over the same field \( \mathbb{F} \), with respective maximal toral subalgebras \( H, H' \) and corresponding root systems \( \Phi \) and \( \Phi' \). Suppose there is an isomorphism between the two root systems (denoted by \( \alpha \mapsto \alpha' \)), inducing \( \pi : H \to H' \). Fix a base \( \Delta \) of \( \Phi \) which then induces a base \( \Delta' \) via this isomorphism. For each simple root \( \alpha \) choose an arbitrary \( x_\alpha \in L_\alpha, x_{\alpha}' \in L_{\alpha}' \) (this means choosing a Lie isomorphism \( \pi_\alpha : L_\alpha \to L_{\alpha}' \) compatible with the root system isomorphism). Then there exists a unique isomorphism \( \pi : L \to L' \) extending the given \( \pi : H \to H' \) and extending all the \( \pi_\alpha \).

This actually says that if two simple Lie algebras have the same Dynkin diagram, then they are isomorphic. This is why we take not all \( \ell \) values for all of the classical Lie algebras, since the classification we have shown wants to clearly separate the different classical Lie algebras. Now the fact that we have can find a simple Lie algebra for any irreducible root system, or then more generally a semisimple Lie algebra for any given root system and the fact that this classification up to isomorphism hold for semisimple Lie algebras as well is given by:

**Existence and Uniqueness Theorems:**
(a) Let \( \Phi \) be a root system. Then there exists a semisimple Lie algebra having \( \Phi \) as its root system.
(b) Let \( L, L' \) be two semisimple Lie algebras over the same field \( \mathbb{F} \), with respective maximal toral subalgebras \( H, H' \) and corresponding root systems \( \Phi \) and \( \Phi' \). Suppose there is an isomorphism between the two root systems (denoted by \( \alpha \mapsto \alpha' \)), inducing \( \pi : H \to H' \). Fix a base \( \Delta \) of \( \Phi \) which then induces a base \( \Delta' \) via this isomorphism. For each simple root \( \alpha \) choose an arbitrary \( x_\alpha \in L_\alpha, x_{\alpha}' \in L_{\alpha}' \) (this means choosing a Lie isomorphism \( \pi_\alpha : L_\alpha \to L_{\alpha}' \) compatible with the root system isomorphism). Then there exists a unique isomorphism \( \pi : L \to L' \) extending the given \( \pi : H \to H' \) and extending all the \( \pi_\alpha \).

So the classification of complex semisimple Lie algebras is unique up to isomorphisms. Since we are usually interested in representations of semisimple Lie algebras rather than in the Lie algebras themselves, it is important to see how they are affected. It is now a fact that if two Lie algebras are isomorphic, then they have the same roots up to an isomorphism. Indeed:

Let \( \phi : L \to L' \) be a Lie algebra isomorphism, with \( H, H' \) the respective maximal toral subalgebras, that are of course isomorphic as well. Then: \( \phi([h, x_\alpha]) = \phi([h, x_{\alpha}']) = \phi(h) \phi(x_\alpha) = \phi(x_{\alpha}') = \alpha'(h') x_{\alpha}' \). So indeed, the roots are isomorphic and this \( \phi : L \to L' \) induces the isomorphism \( \phi^* : H^* \to H'^* \). So the root systems are isomorphic (the isomorphism is the inverse of the one between the two maximal toral subalgebras) and the value of the respective roots for the elements of the respective maximal toral subalgebras are the same.

In conclusion of the classification it is interesting to notice that we merely used the ad-representation. The ad-representation allows us to observe some fundamental properties of a Lie algebra, since it there is actually nothing else than the analysed Lie algebra involved. The ad-representation is thus in a way the characterisation of a Lie algebra and plays an important role in all of the representation theory of Lie algebras as should become clearer in the next section.
iii. Application: Weights

In this section finite dimensional representations of simple Lie algebras are analysed and the objective is to see to what extend the previous classification and other theorems allow us to make predictions.

We define $L$ as a complex simple Lie algebra, $H$ a maximal toral subalgebra and $\phi$ a representation of $L$ onto the vector space $V$. Since $H$ is toral, its representation is diagonalizable and we can do the same thing we did in the case of roots but call them weights $\lambda$. The corresponding $V_\lambda$ are called weight spaces and yield $V$ via a direct sum. The weight space $V_\lambda$ to a given weight $\lambda$ is one dimensional. The roots are actually a special case of weights, the weights of the $ad$-representation. There are now a number of lemmas that we will use and that are more or less intuitive.

**Lemma:** Let $x_\alpha \in L_\alpha$. Let $v \in V_\lambda$. Then $\phi(x_\alpha) v \in V_{\lambda + \alpha}$. The proof is easy and uses the fact that a representation is compatible with the commutator, which is linked to the roots.

**Lemma:** If $\phi$ is an irreducible representation, then there is a unique maximal weight $\mu$ relative to the partial ordering defined for roots. Indeed, we can also find such a base $\Delta_\lambda$ for weights as we did for roots.

Now let $\mu$ be the maximal weight of an irreducible representation $\phi$. Since $\mu$ is maximal, the associated $V_\mu$ is mapped to zero for $\phi(L_\alpha)$ if the root is simple. But we know that $S_\alpha = L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$ is isomorphic to $sl(2, \mathbb{F})$. In other words this means that since the representation of $L_\alpha$ maps $V_\mu$ to zero, that this is the highest weight space of this representation isomorphic to the one of $sl(2, \mathbb{F})$ and that we find all the other weights by applying the corresponding representation of $L_{-\alpha}$. We have seen that the weights are placed symmetrically around zero. But the value zero is associated to the elements of the dual space of $H$ that lie on the hyperplane $P_\alpha$. So this means that the $\alpha$-string through $\lambda$ is symmetric around the hyperplane $P_\alpha$. This symmetric property of the different root strings holds for any roots $\beta \in \Phi$. In other words it is enough to consider the structure of the weights in the fundamental Weyl chamber to the base $\Delta$ of $\Phi$. So now we know the extremal weights of the representation. They form the outer limit of all the possible weights of the representation. The other weights can be found by the roots strings through the different weights $\lambda$, which is obvious since they are obtained by these representations of the $S_\alpha$ for $\alpha \in \Phi$ with a given $0 \prec \lambda$ as highest weight. One can see that the whole lattice of weights that result has 0 in the center of it, the intersection point where all the hyperplanes meet. One can go on and conclude another important fact: since the weights are placed symmetrically around the hyperplane $P_\alpha$ for $\alpha \in \Phi$, this means that $\frac{\langle \lambda \alpha \rangle}{(\alpha, \alpha)}$ is an integer multiple of $1/2$ or in other terms: $\forall \lambda \in \Pi, \forall \alpha \in \Phi : \langle \lambda, \alpha \rangle \in \mathbb{Z}$. Here $\Pi$ is the set of all weights of a representation.

This is a priori the only condition a weight has to satisfy, so it is possible to take closer look of the structure of all possible lattices that can be formed by a given irreducible representation. Define the **fundamental dominant weights** by: $\lambda_i$ satisfies $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$, where $\alpha_j \in \Delta$. These fundamental dominant weights yield all the possible weights in the fundamental Weyl chamber via the sum: $\lambda = \sum_{i=1}^{l} k_i \lambda_i$ where all the $k_i$ are positive. Of course they yield all the a priori possible weights by taking any linear combination. Since the roots are nothing else than weights, the roots are a sublattice of the lattice generated by these fundamental dominant weights. Furthermore, the weights of an irreducible representation are nothing else than a lattice generated by the roots and including a given maximal weight, that means that there are only a finite number of possible
lattices of "different form", exactly as many as there are linear combinations of the fundamental weights which cannot be written as a combination of those fundamental weights and a root. What is meant by the expression "different form" is easier to explain by a few examples. One can easily find that if the highest weight can be written as the linear combination of roots, that 0 is a weight. If not 0 will not be a weight but this given linear combination of fundamental dominant weights, that cannot be decomposed a linear combination containing roots, will be a weight.

So, the classification of simple Lie algebras can be used to describe all the possible representations of simple Lie algebras. In other words, one Dynkin diagram yields only a finite number of possible forms of the irreducible representation which can be used to make more general statements of representations of a given class of complex simple Lie algebras as classified before.

V. KAC-MOODY ALGEBRAS

In this section we will see a way of constructing infinite dimensional Lie algebras that are called affine Kac-Moody algebras. The concept is used in different areas going from the first use in the current algebra approach of local symmetry of elementary particles, string theory, two dimensional
statistical models to two dimensional $\sigma$-models.

i. Construction

The easiest way to introduce the concept of Kac-Moody algebras is to actually consider it as a way to describe local symmetry for a 2-dimensional space-time where the space component is compact, i.e. a circle. The gauge is given by a certain compact finite dimensional connected group $G$ and we will now define a group $\mathcal{G}$, the loop group of $G$ by:

$$\mathcal{G} = \{ \gamma \text{ smooth enough map} | \gamma : S^1 \rightarrow G, x \mapsto \gamma(x) \}$$

the operation on this loop group is given by point by point multiplication:

$$\gamma_1 \circ \gamma_2 = \gamma_1(x) \circ \gamma_2(x).$$

If the Lie algebra of $G$ is given by $\mathfrak{g}$ with $[T^a, T^b] = i f^{ab}_c T^c$, we call the Lie algebra of the loop group the **affine Kac-Moody loop algebra** $\mathfrak{g}_0$. In order to construct it, we use that $G$ is connected and that we can write any element of $G$ via the exponential map using the Lie algebra $\mathfrak{g}$: $x = \exp(-iT^a \theta_a) \in G$ for $\theta_a$ real parameters. In the case of the loop group $\mathcal{G}$ this is:

$$\gamma(z) = \exp(-iT^a \theta_a(z))$$

So near the identity one obtains:

$$\gamma(z) \approx 1 - iT^a \theta_a(z)$$

So now the problem is reduced to a set of maps $\theta_a$. Since we can identify $S^1$ with all $z \in \mathbb{C}$ so that $|z| = 1$. This allows us to expand $\theta(z)$ in the most general way, a Laurent expansion:

$$\sum_{n=-\infty}^{\infty} \theta_{\theta_a} z^n$$

Using this expansion, we set:

$$T^a_n = T^a z^n$$
Thus one obtains:
\[ \gamma(z) \approx 1 - i \sum_{n=-\infty}^{\infty} T_n^a \theta_n^a \tag{5} \]

These are the generators of a obviously infinite dimensional Lie algebra, the affine Kac-Moody loop algebra \( \mathfrak{g}_0 \) given by:
\[ [T_m^a, T_n^b] = i f_{\ c}^{\ ab} T_c^{m+n} \tag{6} \]

ii. Central Extension

Now that we have found the affine Kac-Moody loop algebra \( \mathfrak{g}_0 \) we can go a step further and try to find its central extension. Extending a Lie algebra \( L \) means adding new elements to it but in a way that the new \( L' \) is a Lie algebra as well. A central extension is then extending \( L \) by central elements \( k^j \) that commute with any other element of \( L \) and commute with themselves. Formally:
\[ [T^a, T^b] = i f_{\ c}^{\ ab} T_c^c + id^{ab} j^j \tag{7} \]
\[ [T^i, k^j] = [k^i, k^j] = 0 \tag{8} \]

The reason why this is done can be found in Dirac’s quantization procedure:
\[ [T^a, T^b] = i \hbar f_{\ c}^{\ ab} T_c^c + O(\hbar^2) \tag{9} \]

So in a central extended Lie algebra the central elements play the role of the second order term but do not change the underlying Lie algebra to a great extend. This is why it is an easy way to have a look at the second order term.

Now we will actually extend a Lie algebra \( L \) with the commutation relations as seen in eq.7. Since it still is a Lie algebra this means that the extended Lie algebra \( L' \) has to satisfy:
\[ [[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 \tag{10} \]

Which is the Jacobi identity and translates into:
\[ f_{\ d}^{\ ab} d^{dc} j^j + f_{\ d}^{\ bc} d^{da} j^j + f_{\ d}^{\ ca} d^{db} j^j = 0 \tag{11} \]

And the antisymmetry:
\[ d^{ab} j^j = -d^{ba} j^j \tag{12} \]

One can redefine the generator in the following way:
\[ T^a \rightarrow T^a - \zeta^a j^j \tag{13} \]
Which leads to:
\[ d^{ab} j^j \rightarrow d^{ab} j^j + i f_{\ c}^{\ ab} \zeta^c \tag{14} \]

This redefinition leads to some \( d^{ab} j^j \) becoming trivial. It corresponds to a term of order \( \hbar \) and is very small in the classical limit. The interesting solutions for the central elements are thus the solutions modulo this redefinition.

To present the result it is important make a distinction between the finite dimensional Lie algebras we used to construct the Kac-Moody algebras and the Kac-Moody algebras. In the case of the Lie algebra \( \mathfrak{g} \) the central elements are all trivial and the central extended Lie algebra is thus just \( \mathfrak{g} \).

In the case of the Kac-Moody algebras, we can find nontrivial central elements and obtain the result:
\[ [T_m^a, T_n^b] = i f_{\ c}^{\ ab} T_{m+n}^c + km \delta_{a,b} \delta_{m,-n} \tag{15} \]

It is important to notice that when actually using the extended affine Kac-Moody algebra \( \mathfrak{g} \) in representations, additional conditions for \( k \) occur and these quantise the central elements.
iii. Dynkin diagrams

A classification based on the previous one can be achieved and the corresponding Dynkin diagrams are the following:

![Dynkin Diagrams](image)

*Figure 7: All the possible Dynkin diagrams for affine Kac-Moody algebras. Image taken from Wikipedia*

The additional vertex shows up since the algebra is extended by central elements. In general it is a nontrivial task to find these diagrams even though we know them in the case of the underlying Lie algebras. The reason is the infinite dimensionality of the extended affine Kac-Moody algebra having infinite dimensional maximal toral subalgebras. For the Dynkin diagrams the choice of using the a maximal toral subalgebra of the underlying Lie algebra is important, but is not a maximal toral subalgebra of $G$. These and many other things make it way harder to obtain the shown result.

VI. Appendix

i. Lie algebras

The most general definition of a **Lie algebra** is that of a vector space $L$ over a field $F$ with a operation $L \times L \to L$, $(x, y) \mapsto [x, y]$ called a Lie bracket. The Lie bracket is:

a) bilinear
b) antisymmetric: $\forall x \in L : [x, x] = 0$

A Lie algebra isomorphism is defined as a vector space $\phi$ isomorphism between two Lie algebras $L, L'$ if the following relations holds:

$[\phi(x), \phi(y)]_{L'} = \phi([x, y]_L)$

This definition does not acknowledge the connection between a Lie group $G$ and the corresponding Lie algebra $\mathfrak{g}$. Indeed, $\mathfrak{g}$ can be described as the tangent space to the Lie group in the identity. The adjoint representation $G$ then defines in a natural way a Lie bracket called the commutator $[x, y]=xy-yx$ on $\mathfrak{g}$, the adjoint representation of a Lie algebra. Because of this fact it is possible to map the $\mathfrak{g}$ onto $G$ via the exponential map defined as:

$\exp: \mathfrak{g} \times \mathbb{R} \to G, \exp(x, t)=e^{tx} \in G$

If the group $G$ is connected, the exponential map has as image the whole group $G$.

Call $S$ a **(Lie) subalgebra**, if $S$ is a subspace of a a Lie algebra $L$, that is closed under the Lie bracket. $S$ is a Lie algebra in its own right inheriting the Lie bracket of $L$.

A **Lie algebra isomorphism** is defined as a vector space $\phi$ isomorphism between two Lie algebras $L, L'$ if the following relations holds:

$[\phi(x), \phi(y)]_{L'} = \phi([x, y]_L)$

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We then say, that $L, L'$ are isomorphic, if such an isomorphism exists.

We call a representation of Lie algebras an vector space homomorphism $\phi: L \to \mathfrak{gl}(V)$ for which the relation $\phi([x, y]_L) = [\phi(x), \phi(y)]_{\mathfrak{gl}(V)}$ holds.

We say a representation $\phi$ on $V$ is irreducible if there is no other subspace $W$ than 0 and $V$, so that $\forall x \in L: \phi(x)W \subset W$. If the representation allows us to decompose $V = V_1 \oplus \ldots \oplus V_n$ ($n \leq \dim V$) so that the representation $\phi$ is irreducible on those $V_i$, we call the representation completely reducible.

An ideal $I$ of a Lie algebra $L$ is a subalgebra that satisfies the following condition:
$\forall i \in I \forall x \in L: [i, x] \in I$

The Killing form $\kappa$ is a symmetric bilinear form defined by:
$\kappa: L \times L \to \mathbb{F}, \kappa(x, y) = \text{tr}(\text{ad}(x)\text{ad}(y))$

A nondegenerate bilinear from $F$ is a bilinear from which has a trivial radical $S$. This means:
$S = \{ x \in L | \forall y \in L: F(x, y) = 0 \} = 0$

The centralizer $C_L(X)$ of a subset $X$ is given as:
$C_L(X) = \{ y \in L | \forall x \in X: [x, y] = 0 \}$

The normalizer of a subalgebra $K$ of $L$ is defined by:
$N_L(K) = \{ x \in L | [x, K] \subset K \}$. If $K = N_L(K)$, we call $K$ self-normalizing.

A Cartan subalgebra of a Lie algebras is a nilpotent subalgebra which equals its normalizer in $L$.

Proposition: If $F$ algebraically closed, then the maximal toral subalgebra and the Cartan subalgebra of a complex simple Lie algebras are the same.

ii. $\mathfrak{sl}(2, \mathbb{F})$ representations

The ad-representation on $\mathfrak{sl}(2, \mathbb{F})$ looks like this:
If $x = E_{1,2}, y = E_{2,1}, h = [x, y] = H_1$:
$[h, x] = 2x, [h, y] = -2y$

If we take a given representation $\phi$ of $L$ on $V$, and since $h$ is semisimple, $\phi(h)$ is semisimple, too. This leads to a decomposition of $V$ into eigenspaces $V_\lambda$, the spaces spanned by eigenvectors of $\phi(h)$ to the eigenvalue $\lambda$. Call those eigenvalues weights of the representation, the eigenspaces associated to a weight weight spaces.

Lemma:
If $v \in V_\lambda$, then $\phi(x)v \in V_{\lambda+2}$ and $\phi(y)v \in V_{\lambda-2}$.

The proof uses the commutation relations and the fact that a representation is compatible with the Lie bracket.

Since we usually assume $V$ to be finite dimensional, this leads to $\phi(x)$ and $\phi(y)$ to be nilpotent endomorphisms. If now $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$, call an element of $V_{\lambda+2}$ a maximal vector of weight $\lambda$. 

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For an irreducible representation $\phi$ of $sl(2,F)$ on $V$, choose a maximal vector $v_0 \in V_{\lambda}$, set $v_{-1} = 0$ and $v_i = (1/i!)\phi(y)^i v_0$ for $i \geq 0$, obtain

**Lemma:**

(a) $\phi(h)v_i = (\lambda - 2i)v_i$
(b) $\phi(y)v_i = (i + 1)v_{i+1}$
(c) $\phi(x)v_i = (\lambda - i + 1)v_{i-1}$

Since $V$ is finite dimensional by assumption, there may not be an infinite amount of weight spaces. Since we assume that the representation is irreducible, there is no weight space that is left out in terms of going through them as in the lemma. This means that $\phi(y)$ is nilpotent, just as stated before. So there is an $i = m + 1$ such that $\lambda = m$, which has to be a nonnegative integer, since we go in steps of 1. So there is not only an upper bound for the weight spaces $V_{\lambda}$ but a lower one as well. It turns out that the highest weight of $V$ is equal to the number of times one can apply $\phi(y)$ to $v_0$ before it one obtains zero. We call this $\lambda$ the **highest weight** of $V$. In an irreducible representation every weight occurs once. All this is given in

**Theorem:**

$V$ irreducible, $L=sl(2,F)$

(a) Relative to $\phi(h)$, $V$ is the direct sum of weight spaces $V_{\mu}$ with $\mu = m, m - 2, ..., -(m - 2), -m$, where $m+1= \dim V$ and $\dim V_{\mu}=1$ for each $\mu$.
(b) $V$ has (up to nonzero scalar multiples) a unique maximal vector, whose weight is $m$.
(c) The representation of $L$ on $V$ is given by the above formulas, if the basis is chosen in the prescribed fashion. In particular, there exists at most one irreducible representation (up to isomorphism) of each possible dimension $m+1, m \geq 0$.

This means if we have an irreducible representation of $L \cong sl(2,F)$ on $V$, this completely describes how it behaves. The corollary for any finite dimensional $V$ might be useful:

**Corollary:**

Let $(\phi,V)$ be any finite dimensional representation of $L$, as before. Then the eigenvalues of $\phi(h)$ on $V$ are all integers, and each occurs along with its negative (an equal number of times). Moreover, in any decomposition of $V$ into a direct sum of irreducible representations, the number of summands is precisely $\dim V_0 + \dim V_1$.

This says that any representation is made up of irreducible representations and the number of irreducible representations is given by how often we obtain 0 and 1, since any irreducible representation has these values as weights once, but not both.

**References**

[1] Humphreys, J. E. 1972 Introduction to Lie Algebras and Representation Theory