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An Introduction to Supersymmetry

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1 Introduction

During the course of this paper I will try to give an overview of what supersymmetry (SUSY) is and how to work with the mathematical structures that it contains. The next two chapters will introduce the necessary mathematical tools, which we require for chapters four through six. Some sections, such as 3.3, 3.4, and 3.6 are not necessary to understand later chapters, but give further insight into the physical interpretation of these mathematical constructs and introduce some notation that will be used afterwards.
This chapter is based on [1] and [2]. Without going too much into detail, I will give a short introduction as to what SUSY is. Up until the 1960s, attempts were made to combine all known symmetries into a single group. It wasn't until Coleman and Mandula released their famous "no-go" theorem, that it was realized all such efforts would be in vain. This theorem states that the most general bosonic symmetry can be written as a direct product of the Poincare group and internal symmetries [1], i.e.

$$G = G_{\text{Poincare}} \times G_{\text{Internal}} \ . \tag{1}$$

However, the Coleman-Mandula theorem has a loophole. In 1975, Haag, Lopuskanski, and Sohnius managed to extend the theorem to include fermionic symmetry generators, which are given by the letter Q, that map bosons to fermions and vice-versa. As a result, it turns out that the most general symmetry group can be written as the direct product of the super-Poincare group, which is given by the Poincare generators and the fermionic generators Q, and internal symmetries [1]. We therefore get that

$$G = G_{\text{super-Poincare}} \times G_{\text{Internal}} \ . \tag{2}$$

Let's look at transformation properties of Q. If we apply a unitary operator U that represents a full rotation of 360°, we get that

$$UQ|\text{boson}\rangle = \mathrm{UQU^{-1}U}|\text{boson}\rangle = \mathrm{U}|\text{fermion}\rangle \tag{3}$$
$$UQ|\text{fermion}\rangle = \mathrm{UQU^{-1}U}|\text{fermion}\rangle = \mathrm{U}|\text{boson}\rangle \ . \tag{4}$$

We also know that under a 360° rotation, the two states transform as

$$U|\text{fermion}\rangle = -|\text{fermion}\rangle \tag{5}$$
$$U|\text{boson}\rangle = |\text{boson}\rangle \ . \tag{6}$$

Since the sum of all fermionic and bosonic states form the basis of our Hilbert space, we get that

$$UQU^{-1} = -Q \ , \tag{7}$$

which transforms exactly like a fermionic state [2]. During the course of chapter 4 we will calculate the anticommutator $\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q$. It should be noted that while the Qs will not be unitary in general, $\{Q, Q^\dagger\}$ is. In addition, the anticommutator will have positive eigenvalues, since

$$\langle ...|QQ^\dagger|...\rangle + \langle ...|Q^\dagger Q|...\rangle = |Q^\dagger|...\rangle|^2 + |Q|...\rangle|^2 \geq 0 \ . \tag{8}$$

A further analysis will show that this anticommutator can be written as a linear combination of the energy and momentum operators

$$\{Q, Q^\dagger\} = \alpha E + \vec{\beta}\vec{P} \ . \tag{9}$$

In addition, the sum over all the Qs is proportional to the energy operator, as all other terms will cancel out

$$\sum_{\text{all Q}} \{Q, Q^\dagger\} \propto E \ . \tag{10}$$

Since the left-hand-side is positive, the energy spectrum can either be only positive or negative. If, for physical reasons, we require energies bounded from below, the proportionality factor is positive. This would then require our energy values to be non-negative.

In chapters 4 and 6, we will analyze the representations of the super-Poincare group, which correspond to families of particles. Since any supersymmetry transformation will turn a particle into something else, these families will always contain more than one particle [2]. This tells us that all families, which are also called supermulitplets, must contain at least one boson and one fermion.

Before we start with section 2, there is one more thing I would like to mention that will not be discussed in later chapters. As we will see, our Qs will commute with the energy and momentum operators. This in turn, means that Q changes neither the energy nor the momentum of the particle and therefore all states in the supermultiplet must have the same mass. However, we do not see this in nature, which tells us that either supersymmetry isn't fundamental to nature or there is spontaneously broken symmetry, which means that our ground state will not be invariant under SUSY transformations. Spontaneously broken symmetry would lift this mass degeneracy [2].

2 Lorentz and Poincare Groups and their Algebras

This chapter is based on [3] and [4].

2.1 Lorentz Group

The Lorentz group $O(1,3)$ is given by all linear transformations in $\mathbb{R}^4$ that conserve the Minkowski metric. In other words,

$$O(1,3) = \{A \in GL(4,\mathbb{R}) | (Ax, Ax) = (x,x), \forall x \in \mathbb{R}^4\} \ , \tag{11}$$

where $(x,y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$ and $x, y \in \mathbb{R}^4$. If we write A, where $A \in O(1,3)$, as a matrix, this can be interpreted as follows:

$$A^T g A = g, \text{where } g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{12}$$

While these matrices have a great many number of properties, we are only interested in a handful of them. In the following, I will make certain claims and prove them as well [3].
Claim 1: $det(A) = \pm 1$
Proof:

$$det(A^T gA) = det(A^T)det(g)det(A) = det(g) \ , \tag{13}$$

which implies that $det(A) = \pm 1$.
Claim 2: $(A_{00})^2 \geq 1$
Proof:

$$A^T gA = g \ , \tag{14}$$

where $A = (A_{ij})$ implies that

$$\sum_{k,l} A_{ik} g_{kl} A_{jl} = g_{ij} \ . \tag{15}$$

For $i = j = 0$, we now have

$$\sum_{k,l} A_{0k} g_{kl} A_{0l} = g_{00} \ . \tag{16}$$

We also know that $g_{kl} = 0$ for $k \neq l$. This relationship therefore allows us to remove one of the indices, which gives us

$$A_{00}^2 - A_{01}^2 - A_{02}^2 - A_{03}^2 = 1 \ . \tag{17}$$

Since all entries of A are real, their squares must be greater than or equal to zero. This, in turn, proves that $A_{00}^2 \geq 1$.
We will use these two claims to split $O(1,3)$ into four classes. But before that, we will examine some elements of the Lorentz group [3].
Orthogonal Transformations in $\mathbb{R}^3$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \tag{18}$$

where $R \in O(3)$ is an orthogonal transformation. These matrices form a subroup.
Lorentz Boosts:

$$L(\chi) = \begin{pmatrix} \cosh(\chi) & 0 & 0 & \sinh(\chi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\chi) & 0 & 0 & \cosh(\chi) \end{pmatrix} \tag{19}$$

This matrix represents a Lorentz boost in the z-direction with rapidity $\chi \in \mathbb{R}$. Lorentz boosts in the x- or y-direction can be constructed analogously. These matrices are used in special relativity to change from one inertial system to another. One of their more important properties is that $L(\chi_1)L(\chi_2) = L(\chi_1 + \chi_2)$.
Space Inversion:

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{20}$$

Time Inversion:

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{21}$$

These two discrete Lorentz transformations, along with 1 and PT, generate an Abelian group of order 4.
We will now define two additional subgroups of O(1,3), of which the latter will help us create the four classes.

$$O_+(1,3) = \{A \in O(1,3) | A_{00} \geq 1\} \tag{22}$$

$$SO_+(1,3) = \{A \in O_+(1,3) | det(A) = 1\} \tag{23}$$

In the last paragraph I have ever so slightly made a proposition, but have not proven it. I have namely stated that both of these sets are subgroups, which beforehand does not necessarily have to be true. The following two proofs will do just that[3].
Claim 3: $O_+(1,3)$ is a subgroup of $O(1,3)$. More specifically, it is the set of Lorentz transformations that maps the vectors inside the future light cone(Z_+) to vectors inside the future light cone(Z_+).
Proof:
We will prove the second statement first. Let $A \in O_+(1,3)$. If $x \in Z_+$ then Ax must lie within the light cone, since $0 \leq (x,x) = (Ax, Ax)$. We must now show that Ax indeed lies in the future light cone, i.e. $(Ax)_0 > 0$. We will now introduce the following notation: $\overrightarrow{A_0} = (A_{01}, A_{02}, A_{03})$ and $\vec{x} = (x^1, x^2, x^3)$. The 0-component of Ax will then be:

$$x'^0 = A_{00}x^0 + \overrightarrow{A_0} \cdot \vec{x} \tag{24}$$

Now since $(a,a) = 1$, where a is the first row of A(this will not be proven, but can easily be shown by checking that A^T is a Lorentz transformation and then using definition (12)), a must lie within the light cone. This, in turn, implies that $|\overrightarrow{A_0}| < A_{00}$. By using this, and the fact that $|\vec{x}| < x^0$, we obtain the following relation:

$$x'^0 \geq A_{00}x^0 - |\overrightarrow{A_0}||\vec{x}| > 0 \tag{25}$$

We have thus shown that $Ax \in Z_+$. All that remains to be proven for the second part of the proposition is that Lorentz transformations A, which map Z_+ to Z_+, are elements of $O_+(1,3)$. This transformation must map $(1,0,0,0)^T$ onto a vector in Z_+, i.e. the 0-component must be positive. This, in turn, means that $A_{00} > 0$.
Having finally proven the second part, all that is left to show is that $O_+(1,3)$ is a subgroup. It is clear that a product of transformations from Z_+ to Z_+ must once again be a mapping from Z_+ to Z_+. We must now only prove that if $A \in O_+(1,3)$, so must also be $A^{-1} \in O_+(1,3)$.
Since $A(-x) = -Ax$, A maps the set $Z_- = -Z_+$ onto itself. Given that $A \in O(1,3)$, it maps $Z = Z_- \cup Z_+$ bijectively onto itself. It thusly follows that $A^{-1} \in O_+(1,3)$[3].
Claim 4: $SO_+(1,3)$ is a subgroup of $O(1,3)$.
Proof:
Since $SO_+(1,3)$ is in fact a subset of $O_+(1,3)$, we actually only need to prove that $SO_+(1,3)$ is a subgroup of $O_+(1,3)$. Once again, we must show that the product of two elements, as well as the inverse, still lies in $SO_+(1,3)$. Our previous claim tells us that if $A, B \in SO_+(1,3)$, then $AB \in O_+(1,3)$. We must now only show that $det(AB) = 1$.

$$det(AB) = det(A)det(B) = 1 \cdot 1 = 1 \tag{26}$$

To show that $A^{-1} \in SO_+(1,3)$, we can once again take advantage of our previous claim. We know that $A^{-1} \in O_+(1,3)$, but we still must calculate its determinant.

$$AA^{-1} = \mathbb{1} \Rightarrow det(AA^{-1}) = 1 \Rightarrow det(A^{-1}) = det(A)^{-1} = 1 \tag{27}$$

Finally, we can categorize the Lorentz group into four classes[3].
Claim 5: Every Lorentz transformation lies in exactly one of the following classes: $SO_+(1,3)$, $\{PX|X \in SO_+(1,3)\}$, $\{TX|X \in SO_+(1,3)\}$, $\{PTX|X \in SO_+(1,3)\}$
Proof:
Claims 1 & 2 already show that the determinant must either be 1 or -1 and $|A_{00}| \geq 1$. This already gives us the four classes[3]. If $X \in SO_+(1,3)$, then

	$det = 1$	$det = -1$
$A_{00} \geq 1$	X	PX
$A_{00} \leq -1$	PTX	TX

2.2 Lie Algebra

Definition:
A one-parameter group is a mapping $\mathbb{R} \to GL(n, \mathbb{K})$, $t \mapsto X(t)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, if it is continuously differentiable, $X(0) = \mathbb{1}$, and $\forall t, s \in \mathbb{R}$ $X(s+t) = X(s)X(t)$.
Claim 6: $\forall X \in Mat(n, \mathbb{K})$, $t \mapsto exp(tX)$ is a one-parameter group.
Proof:
We know that $tXsX = sXtX$, which then leads to $exp(tX)exp(sX) = exp(tX + sX)$. $exp(tX)$ is clearly continuously differentiable since $\frac{d}{dx}exp(tX) = exp(tX)X$. For $t = 0$, we get the identity.
Definition:

If a one-parameter group is given by $t \mapsto exp(tX)$, then X is called the (infinitessimal) generator.
Claim 7: All one-parameter groups are given by $t \mapsto exp(tX)$.
Proof:
We know that if $t \mapsto X(t)$ is a one-parameter group, $X(t)$ must then satisfy the following differential equation:

$$\begin{aligned} \frac{d}{dx} exp(tX) &= \lim_{h \to 0} \frac{X(t+h) - X(t)}{h} = \lim_{h \to 0} \frac{X(t)X(h) - X(t)}{h} && (28) \\ &= X(t) \lim_{h \to 0} \frac{X(h) - 1}{h} = X(t)\dot{X}(0), && (29) \end{aligned}$$

where $X(0) = \mathbb{1}$. Due to the uniqueness of the solution, it follows that:

$$X(t) = exp(tX(0)) \tag{30}$$

Definition:
A (Matrix-)Lie-group is a closed subgroup (i.e. for all convergent series X_j in $GL(n, \mathbb{K})$, $X_j \in G \ \forall j \Rightarrow \lim_{j \to \infty} X_j \in G$) of $GL(n, \mathbb{K})$.
We can then also define

$$Lie(G) = \{X \in Mat(n, \mathbb{K}) | \ exp(tX) \in G \ \forall t \in \mathbb{R}\}. \tag{31}$$

Lie(G) is called the Lie-algebra of the Lie-group G.
Definition:
The commutator of X and $Y \in Mat(n, \mathbb{K})$ is given by:

$$[X, Y] = XY - YX \tag{32}$$

The commutator will then have the following properties:

$$\begin{aligned} &(i) && [\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z], \ where \ \lambda, \mu \in \mathbb{K} && (33) \\ &(ii) && [X, Y] = -[Y, X] && (34) \\ &(iii) && [[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0 && (35) \end{aligned}$$

Definition:
A real or complex vector space g, equipped with a "Lie-bracket" $[\ ,\ ] : g \times g \to g$, which exhibits the properties (i)-(iii), is called a (real or complex) Lie-algebra.
This means, for example, that $Lie(G)$ has the structure of a real Lie-algebra.

2.3 Lorentz Algebra

For the rest of the chapter we will be using the Einstein summation convention. Having finally discussed the Lorentz group, as well as Lie algebrae, we can finally construct the Lorentz algebra. Since all elements of the Lorentz algebra can generate a one-parameter group that is connected to the identity, we expect that only the connected space that contains the unity to play a role. This means that only $SO_+(1,3)$ is of importance to us. Since it can be shown (we will not prove this) that $SO_+(1,3)$ only contains rotations, as well as boosts, we should expect six generators that will span the algebra[4]. The previous chapter implies that each $A \in SO_+(1,3)$ that is close to the identity can be written as:

$$A = \mathbb{1} + \omega_{\rho\sigma} J^{\rho\sigma} + ... \ , \tag{36}$$

where the $\omega_{\rho\sigma}$ are real parameters and $J^{\rho\sigma}$ are the generators. In quantum mechanics however, there is a different preferred notation, which does not change the theory, but creates Hermitian generators.

$$A = \mathbb{1} + \frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma} + ... \ . \tag{37}$$

We will use equation (37) from now on. We will now define transformations close to the identity with the help of $\omega_{\rho\sigma}$ and then use this to find our generators.

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu} \tag{38}$$

By applying the definition of elements in the Lorentz group, we get

$$\begin{aligned} \eta_{\rho\sigma} &= \eta_{\mu\nu}(\delta^\mu_{\ \rho} + \omega^\mu_{\ \rho})(\delta^\nu_{\ \sigma} + \omega^\nu_{\ \sigma]}) && (39) \\ &= \eta_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + O(\omega^2) \ . && (40) \end{aligned}$$

By only keeping the linear terms, we see that this is simply an anti-symmetry condition for $\omega_{\mu\nu}$[4].

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{41}$$

Since $\mu, \nu = 0, ..., 3$, the anti-symmetry tells us that there are only six independent components, which is what we were expecting. We can now find the six generators that go along with these six components. Since $\omega_{\rho\sigma}$ is anti-symmetric, we can choose $J^{\rho\sigma}$ to be anti-symmetric as well. By comparing (37) to (38), we get that

$$\begin{pmatrix} 1 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 1 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 1 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 1 \end{pmatrix} = \mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} \ . \tag{42}$$

This tells us that

$$\begin{aligned} J^{01} &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J^{02} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J^{03} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\ J^{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} J^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \ . \end{aligned}$$

Now that we have the generators, we can find the commutation relations.

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \tag{43}$$

We can turn this complicated relation into three simpler commutation relations by separating the generators into rotation and boost generators. We will group the three rotation generators in the angular-momentum three-vector[4]

$$\mathbf{J} = \{J^{23}, J^{31}, J^{12}\} \tag{44}$$

and the boost generators in the "boost" three-vector

$$\mathbf{K} = \{J^{01}, J^{02}, J^{03}\} \ . \tag{45}$$

As a result, we now get

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{46}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k \tag{47}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k \ . \tag{48}$$

2.4 Poincare Group

The Poincare group describes the symmetries of the Minkowski space and is given by the set of transformations of the form:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_{\ \nu}x^\nu + a^\nu \tag{49}$$

The $\Lambda^\mu_{\ \nu}$ are the Lorentz transformations, which we have just seen, whereas the a^ν parameterizes translations. Elements of the Poincare group can be written as (Λ, a), where $(\Lambda, 0)$ would purely be a Lorentz transformation and $(\mathbb{1}, a)$ would be a pure translation. One can easily verify that the Poincare group is a semi-direct product of the Lorentz and 4-translation groups[4], i.e.

$$(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_2 a_1 + a_2) \ . \tag{50}$$

2.5 Poincare Algebra

Similar to chapter 2.3, we will study transformations close to the identity.

$$U(\Lambda, a) = \mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} - i\epsilon_\rho P^\rho + ... \ , \tag{51}$$

where $\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}$ and $a^\mu = \epsilon^\mu$. Once again, the $"i"'s$ are there to make the operators Hermitian. By expanding the following equation to the first order, we can then find necessary relations:

$$U(\Lambda, a)U(\mathbb{1} + \omega, \epsilon)U^{-1}(\Lambda, a) = U(\Lambda(\mathbb{1} + \omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \tag{52}$$

This then namely leads to

$$U(\Lambda,a)[\frac{1}{2}\omega_{\rho\sigma}J^{\rho\sigma}-\epsilon_\rho P^\rho]U^{-1}(\Lambda,a)=\frac{1}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu}-(\Lambda\epsilon-\Lambda\omega\Lambda^{-1}a)_\mu P^\mu\ . \tag{53}$$

By equating the coefficients of $\omega_{\rho\sigma}$ and ϵ_ρ, we thusly get

$$\begin{aligned} U(\Lambda,a)J^{\rho\sigma}U^{-1}(\Lambda,a) &= \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}(J^{\mu\nu}-a^\mu P^\nu+a^\nu P^\mu) & (54)\\ U(\Lambda,a)P^\rho U^{-1}(\Lambda,a) &= \Lambda_\mu^{\ \rho}P^\mu\ , & (55)\end{aligned}$$

which tells you that for homogenous Lorentz transformations, i.e. $a^\mu=0$, $J^{\mu\nu}$ transforms as a tensor and P^ν as a vector. By applying these two rules to an infinitessimal transformation, i.e. $\Lambda^\mu_{\ \nu}=\delta^\mu_{\ \nu}+\omega^\mu_{\ \nu}$ and $a^\mu=\epsilon^\mu$, and expanding everything to first order in $\omega^\mu_{\ \nu}$ and ϵ^μ, we get

$$\begin{aligned} i[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\epsilon_\mu P^\mu, J^{\rho\sigma}] &= \omega_\mu^{\ \rho}J^{\mu\sigma}+\omega_\nu^{\ \sigma}J^{\rho\nu}-\epsilon^\rho P^\sigma+\epsilon^\sigma P^\rho & (56)\\ i[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\epsilon_\mu P^\mu, P^\rho] &= \omega_\mu^{\ \rho}P^\mu\ . & (57)\end{aligned}$$

By once again equating the coefficients of $\omega^\mu_{\ \nu}$ and ϵ^μ, we get the following commutation relations[4]

$$\begin{aligned} i\,[J^{\mu\nu},J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma}-\eta^{\mu\rho}J^{\nu\sigma}-\eta^{\sigma\mu}J^{\rho\nu}+\eta^{\sigma\nu}J^{\rho\mu} & (58)\\ i\,[P^\mu,J^{\rho\sigma}] &= \eta^{\mu\rho}P^\sigma-\eta^{\mu\sigma}P^\rho & (59)\\ [P^\mu,P^\rho] &= 0\ . & (60)\end{aligned}$$

In quantum mechanics, these 10 operators play a very important role. P^1, P^2, and P^3 will be the momentum operator components, whereas P^0 will play the role of the Hamiltonian. We have already discussed the role of the other six generators in chapter 2.3. By renaming $P^0=H$ and once again grouping the other generators into a momentum three-vector $\mathbf{P}$, an angular momentum three-vector $\mathbf{J}$, and a "boost" three-vector $\mathbf{K}$:

$$\begin{aligned} \mathbf{P} &= \{P^1,P^2,P^3\} & (61)\\ \mathbf{J} &= \{J^{23},J^{31},J^{12}\} & (62)\\ \mathbf{K} &= \{J^{01},J^{02},J^{03}\}\ , & (63)\end{aligned}$$

we can simplify the commutation relations into

$$\begin{aligned} [J_i,J_j] &= i\epsilon_{ijk}J_k & (64)\\ [J_i,K_j] &= i\epsilon_{ijk}K_k & (65)\\ [K_i,K_j] &= -i\epsilon_{ijk}J_k & (66)\\ [J_i,P_j] &= i\epsilon_{ijk}P_k & (67)\\ [K_i,P_j] &= -iH\delta_{ij} & (68)\\ [J_i,H] &= [P_i,H]=[H,H]=0 & (69)\\ [K_i,H] &= -iP_i\ . & (70)\end{aligned}$$

As a side note, the Lie algebras discussed in chapter 2 are the same for the representations of the Lorentz and Poincare group. That is why I have used the same notation for the generators that many authors use for the generators of the representation. I will employ the same letters for them as well, but it should be clear as to how I am using them.

3 Spinors

This chapter is based on [5], [6], and [7].

3.1 Introduction to the Spinor Representation

We have discussed the Lorentz transformations in detail and now know their commutation relations, but are there other matrices that solve the Lorentz algebra? Yes! We will do this by explicitly constructing the spinor representation[5]. To start off the search for these matrices, we will define the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}\equiv\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\eta^{\mu\nu}\mathbb{1}\ , \tag{71}$$

where γ^μ, with $\mu = 0, ..., 3$, are a set of four matrices. This means that

$$\begin{aligned}\gamma^\mu\gamma^\nu &= -\gamma^\nu\gamma^\mu \text{ when } \mu \neq \nu &(72)\\ (\gamma^0)^2 &= \mathbb{1} &(73)\\ (\gamma^i)^2 &= -\mathbb{1} \text{ for i} = 1,2,3\ . &(74)\end{aligned}$$

It can easily be shown that 2×2 or 3×3 matrices cannot satisfy these conditions. So, the simplest representation of the Clifford algebra is in terms of 4×4 matrices. Four such matrices that fulfill the Clifford algebra are

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{75}$$

where each element is given by a 2×2 matrix and σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{76}$$

Now we have found four matrices, but we require six, since that is the number of generators the Lorentz transformation has. We will now construct six matrices out of these four by using the commutator[5].

$$S^{\rho\sigma} = -\frac{i}{4}[\gamma^\rho, \gamma^\sigma] = \left\{ \begin{array}{cc} 0 & \rho = \sigma \\ -\frac{i}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{array} \right\} = -\frac{i}{2}\gamma^\rho\gamma^\sigma + \frac{i}{2}\eta^{\rho\sigma} \tag{77}$$

Claim 8: $[S^{\mu\nu}, \gamma^\rho] = -i\gamma^\mu\eta^{\nu\rho} + i\gamma^\nu\eta^{\rho\mu}$

Proof:

Without loss of generality we can set $\mu \neq \nu$ and obtain

$$\begin{aligned}[S^{\mu\nu}, \gamma^\rho] &= -\frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] &(78)\\ &= -\frac{i}{2}\gamma^\mu\gamma^\nu\gamma^\rho + \frac{i}{2}\gamma^\rho\gamma^\mu\gamma^\nu &(79)\\ &= -\frac{i}{2}\gamma^\mu\{\gamma^\nu, \gamma^\rho\} + \frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu + \frac{i}{2}\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \frac{i}{2}\gamma^\mu\gamma^\rho\gamma^\nu &(80)\\ &= -i\gamma^\mu\eta^{\nu\rho} + i\gamma^\nu\eta^{\rho\mu}\ . &(81)\end{aligned}$$

Claim 9: The matrices $S^{\mu\nu}$ form a representation that satisfies the commutation relation of the Lorentz algebra, which is given by

$$i\,[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} - \eta^{\sigma\mu}S^{\rho\nu} + \eta^{\sigma\nu}S^{\rho\mu}\ . \tag{82}$$

Proof:

By setting $\rho \neq \sigma$ and using claim 8, we get

$$\begin{aligned}i\,[S^{\mu\nu}, S^{\mu\nu}] &= \frac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] &(83)\\ &= \frac{1}{2}[S^{\mu\nu}, \gamma^\rho]\,\gamma^\sigma + \frac{1}{2}\gamma^\rho\,[S^{\mu\nu}, \gamma^\sigma] &(84)\\ &= -\frac{i}{2}\gamma^\mu\gamma^\sigma\eta^{\nu\rho} + \frac{i}{2}\gamma^\nu\gamma^\sigma\eta^{\rho\nu} - \frac{i}{2}\gamma^\rho\gamma^\mu\eta^{\nu\sigma} + \frac{i}{2}\gamma^\rho\gamma^\nu\eta^{\sigma\mu}\ . &(85)\end{aligned}$$

By now taking equation (77), which gives us that $\gamma^\mu\gamma^\sigma = 2iS^{\mu\sigma} + \eta^{\mu\sigma}$, we get

$$i\,[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} - \eta^{\sigma\mu}S^{\rho\nu} + \eta^{\sigma\nu}S^{\rho\mu}\ . \tag{86}$$

3.2 Spinors

Having finally found six 4×4 matrices that fulfill the Lorentz algebra, we now need a field on which they act upon. Here, we call this field the Dirac spinor field $\psi^\alpha(x)$[5]. We require $\psi^\alpha(x)$ to have four complex components, which we will by $\alpha = 1,2,3,4$. For clarity, we will also index the rows and columns of our 4×4 matrices with $\alpha, \beta = 1,2,3,4$. So, under Lorentz transformations, we have

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta\psi^\beta(\Lambda^{-1}x)\ , \tag{87}$$

where

$$\Lambda = exp\left(\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right) \tag{88}$$

$$S[\Lambda] = exp\left(\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\ . \tag{89}$$

Whilst the generators $J^{\rho\sigma}$ and $S^{\rho\sigma}$ are different, we still use $\omega_{\rho\sigma}$ in both cases. This makes sure that we are acting on x and ψ equally. With the help of two examples, we will now show how these two representations are indeed different.
Rotations:

$$S^{ij} = -\frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \text{ (for i} \neq \text{j)} \tag{90}$$

By writing the parameters as $\omega_{ij} = -\epsilon_{ijk}\varphi^k$, the rotation matrix is then given by

$$S[\Lambda] = exp\left(\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} . \tag{91}$$

Here, $\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$ and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. Let us now consider a rotation by 2π around the x^3-axis, i.e. $\vec{\varphi} = (0, 0, 2\pi)$. For a vector we would expect the rotation matrix to be the identity, but that is not the case for the spinor rotation matrix

$$S[\Lambda] = \begin{pmatrix} e^{+i\pi\sigma^3} & 0 \\ 0 & e^{+i\pi\sigma^3} \end{pmatrix} = -\mathbb{1} . \tag{92}$$

This means that under a 2π rotation

$$\psi^\alpha \to -\psi^\alpha . \tag{93}$$

Boosts:

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{94}$$

By writing the boost parameters as $\omega_{i0} = \chi_i$, we then get

$$S[\Lambda] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} . \tag{95}$$

Representations of the Lorentz Group are not Unitary
For the rotations, we have $S[\Lambda]^\dagger S[\Lambda] = \mathbb{1}$, which implies that $S[\Lambda]$ is unitary. But for boosts, that is no longer the case. It can be shown that there are in fact no finite dimensional unitary representations of the Lorentz group[5].

3.3 Constructing an Action

Having finally found a new field, the next step would be to find a Lorentz invariant equation of motion. To do this, we will first find a Lorentz invariant action. Looking at the four matrices that fulfill the Clifford algebra, it can easily be shown that $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$, which then leads to the conclusion that for all $\mu = 0, 1, 2, 3$, we have

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger . \tag{96}$$

This then tells us that

$$(S^{\mu\nu})^\dagger = \frac{i}{4}\left[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\right] = -\frac{i}{4}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right] \tag{97}$$

$$= -\frac{i}{4}\left[\gamma^0\gamma^\mu\gamma^0, \gamma^0\gamma^\nu\gamma^0\right] = \gamma^0\left(-\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]\right)\gamma^0 \tag{98}$$

$$= \gamma^0 S^{\mu\nu}\gamma^0 . \tag{99}$$

This identity can be used to obtain the following result

$$S[\Lambda]^\dagger = exp\left(-\frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0 . \tag{100}$$

With this in mind, we now define the Dirac adjoint

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0 . \tag{101}$$

Claim 10: $\overline{\psi}\psi$ is a Lorentz scalar
Proof:
By applying a Lorentz transformation, we get that

$$\overline{\psi}(x)\psi(x) = \psi^\dagger(x)\gamma^0\psi(x) \tag{102}$$

$$\to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) \tag{103}$$

$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x) \tag{104}$$

$$= \overline{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x) , \tag{105}$$

which is exactly how a Lorentz scalar should transform[5].
Claim 11: $\overline{\psi}\gamma^\mu\psi$ is a Lorentz vector, which implies that it transforms as

$$\overline{\psi}(x)\gamma^\mu\psi(x) \to \Lambda^\mu{}_\nu\overline{\psi}(\Lambda^{-1}x)\gamma^\mu\psi(\Lambda^{-1}x) \ . \tag{106}$$

This means that we can treat the $\mu = 0, 1, 2, 3$ as a true vector index.
Proof:
For convenience, we will not explicitly write x. Under a Lorentz transformation, we get

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi \ , \tag{107}$$

which tells us that if $\overline{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector, it must be that

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu \ . \tag{108}$$

This is the equality that we will now show. If we work infinitessimally, we get

$$\Lambda = exp\left(\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right) \approx \mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} + ... \tag{109}$$

$$S[\Lambda] = exp\left(\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) \approx \mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma} + ... \ , \tag{110}$$

which tells us that the requirement (110) turns to

$$-[S^{\rho\sigma}, \gamma^\mu] = (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu \ . \tag{111}$$

We will show this by using Claim 8. The right side of (111) can be written out as

$$(J^{\rho\sigma})^\mu{}_\nu\gamma^\nu = (i\eta^{\sigma\mu}\delta^\rho_\nu - i\eta^{\rho\mu}\delta^\sigma_\nu)\gamma^\nu \tag{112}$$

$$= i\eta^{\sigma\mu}\gamma^\rho - i\eta^{\rho\mu}\gamma^\sigma \ , \tag{113}$$

which then turns (111) into

$$-[S^{\rho\sigma}, \gamma^\mu] = i\eta^{\sigma\mu}\gamma^\rho - i\eta^{\rho\mu}\gamma^\sigma \ , \tag{114}$$

which is exactly what Claim 4 says[5].
Claim 12: $\overline{\psi}\gamma^\mu\gamma^\nu\psi$ is a Lorentz tensor. To be precise, the symmetric part transforms as a Lorentz scalar and is proportional to $\eta^{\mu\nu}\overline{\psi}\psi$, whereas the anti-symmetric part is given by a Lorentz tensor and is proportional to $\overline{\psi}S^{\mu\nu}\psi$.
Proof:
Just like Claim 11.
To build the Lorentz invariant action, we only need $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\mu\psi$.

$$S = \int d^4x \ \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{115}$$

This is known as the Dirac action. As a side note, after quantization, this theory describes particles of mass $|m|$ and spin $\frac{1}{2}$.

3.4 The Dirac Equation

By varying the action given by (115) with respect to ψ and $\overline{\psi}$ independently, we obtain the equations of motion. If we vary with respect to $\overline{\psi}$, we obtain the famous Dirac equation.

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{116}$$

We can then obtain the conjugate equation by varying with respect to ψ.

$$i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0 \tag{117}$$

The Dirac equation is Lorentz invariant, which may be a bit surprising, since it doesn not contain second order (or higher) derivatives. In addition, the Dirac equation mixes up the different components of ψ with the help of the matrices γ^μ. Nevertheless, all of the four components solve the Klein-Gordon equation.

$$(\partial_\mu\partial^\mu + m^2)\psi^\alpha = 0 \tag{118}$$

To show this, we start by writing

$$(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi = -(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi = 0 \ . \tag{119}$$

But since $\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = \frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\partial_\nu\partial_\mu = \partial_\mu\partial^\mu$, we then get

$$-(\partial_\mu\partial^\mu + m^2)\psi = 0 \ . \tag{120}$$

Since there are no matrices present, this equation implies that each component ψ^α fulfills equation (118)[5].
The Slash
To make notation more compact, I will introduce some new notation

$$A_\mu\gamma^\mu = \not{A} \ . \tag{121}$$

The Dirac equation can then be written as

$$(i\not{\partial} - m)\psi = 0 \tag{122}$$

3.5 Chiral Spinors

At the beginning of Chapter 3, we explicitly wrote down four matrices that fulfill the Clifford algebra. As we will see in a later subsection, these are NOT the only possible solution. In our case, we used the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ . \tag{123}$$

In this representation, we ended up with block diagonal spinor transformations

$$S[\Lambda_{rot}] = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} \ S[\Lambda_{boost}] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \ , \tag{124}$$

which means that the Dirac spinor representation is reducible. We can therefore deconstruct these matrices into two irreducible representations, which only act on two-component spinors $u_\pm$. In the chiral representation, these are defined as

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \ . \tag{125}$$

These $u_\pm$ are called Weyl spinors or chiral spinors. It's easy to see that under rotation, both transform as

$$u_\pm \to e^{+i\vec{\varphi}\cdot\vec{\sigma}/2}u_\pm \ , \tag{126}$$

whereas under boosts, we get

$$u_\pm \to e^{\pm\vec{\chi}\cdot\vec{\sigma}/2}u_\pm \ . \tag{127}$$

As we will discuss in Chapter 4, u_+ is in the $(\frac{1}{2},0)$ representation of the Lorentz group, while u_- is in the $(0,\frac{1}{2})$ representation. We can therefore say that the Dirac spinor ψ lies in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation[5].

3.6 Weyl Equation

We will now write the Dirac Lagrangian using Weyl Spinors

$$L = \overline{\psi}(i\not{\partial} - m)\psi = iu_-^\dagger\sigma^\mu\partial_\mu u_- + iu_+^\dagger\overline{\sigma}^\mu\partial_\mu u_+ - m(u_+^\dagger u_- + u_-^\dagger u_+) = 0 \ , \tag{128}$$

where

$$\sigma^\mu = \left(\mathbb{1},\sigma^i\right) \ and \ \overline{\sigma}^\mu = \left(\mathbb{1},-\sigma^i\right) \ . \tag{129}$$

If the fermion has a mass, the last term ensures a coupling between u_+ and u_-. But, if we are working with massless fermions, then they can be described by Weyl spinors. For the latter, we then obtain the equation of motion

$$i\overline{\sigma}^\mu\partial_\mu u_+ = 0 \tag{130}$$

$$\text{or } \mathrm{i}\sigma^\mu\partial_\mu \mathrm{u}_- = 0 \ . \tag{131}$$

These are known as the Weyl equations[5].

3.7 Majorana Fermions

So far, our spinors ψ^α have been complex objects. This is because our representation $S[\Lambda]$ is usually complex. So even if we try to make our spinor ψ real, applying a Lorentz transformation would make it complex again. This means that in order to create real spinors, we would have to find a new basis for the Clifford algebra. Our new four matrices are known as the Majorana basis and are defined by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \tag{132}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} . \tag{133}$$

Not only do they satisfy the Clifford algebra, they are also pure imaginary, i.e. $(\gamma^\mu)^\star = -\gamma^\mu$. This then creates pure imaginary generators, which means that our matrices $S[\Lambda]$ are real. By imposing the condition that

$$\psi = \psi^\star , \tag{134}$$

we can now work with real spinors without worrying that a Lorentz transformation will make it complex. These spinors are called Majorana spinors.
But what do we do if we use a general basis for the Clifford algebra? We only additionally require that our four matrices satisfy $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. We also need to define the charge conjugate of a Dirac spinor ψ,

$$\gamma^{(c)} = C\psi^\star , \tag{135}$$

where C is a 4×4 matrix that fulfills

$$C^\dagger C = \mathbb{1} \textit{ and } C^\dagger \gamma^\mu C = -(\gamma^\mu)^\star . \tag{136}$$

We should now first check that our definition is "good", i.e. $\psi^{(c)}$ transforms nicely under Lorentz transformations.

$$\psi^{(c)} \to CS[\Lambda]^\star \psi^\star = S[\Lambda]C\psi^\star = S[\Lambda]\psi^{(c)} , \tag{137}$$

where we used (136) to take C through $S[\Lambda]$. In addition, $\psi^{(c)}$ also fulfills the Dirac equation, if ψ does.

$$(i\partial\!\!\!/ - m)\psi = 0 \quad \Rightarrow \quad (-i\partial\!\!\!/^\star - m)\psi^\star = 0 \tag{138}$$

$$\Rightarrow \quad C(-i\partial\!\!\!/^\star - m)\psi^\star = (+i\partial\!\!\!/ - m)\psi^{(c)} = 0 \tag{139}$$

The general Lorentz invariant reality condition on a Dirac spinor is then

$$\psi^{(c)} = \psi . \tag{140}$$

As a side note, after quantization, Majorana spinors define fermions that are its own anti-particle.
We will now find the matrix C for two representations of the Clifford algebra. As you may recall, in the Majorana basis, our γ^μ are purely imaginary. This, in turn, gives us $C_{Maj} = \mathbb{1}$ and our Majorana condition $\psi^{(c)} = \psi$ is simplified to $\psi^\star = \psi$. For chiral basis (123), only γ^2 is imaginary, which allows us to take $C_{chiral} = i\gamma^2 = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}$. With this, we can also find how the Majorana condition (134) looks in terms of Weyl spinors. By plugging in all the various definitions, we get that $u_+ = i\sigma^2 u_-^\star$ and $u_- = -i\sigma^2 u_+^\star$[5]. This then gives us

$$\psi = \begin{pmatrix} u_+ \\ -i\sigma^2 u_+^\star \end{pmatrix} . \tag{141}$$

3.8 Weyl Spinors and Invariant Tensors

Using our knowledge of Dirac spinors, we can determine how Weyl spinors transform:

$$\psi_\alpha \to \psi'_\alpha = M_\alpha^{\ \beta}\psi_\beta \quad \alpha, \beta = 1, 2 , \tag{142}$$

where $M \in SL(2, \mathbb{C})$. It is easy to verify that the Dirac adjoint of a Weyl spinor will once again be a Weyl spinor. Using this information, we can then define a barred Weyl spinor, which transforms as

$$\overline{\psi}_{\dot\alpha} \to \overline{\psi'}_{\dot\alpha} = (M^\star)_{\dot\alpha}^{\ \dot\beta}\overline{\psi}_{\dot\beta} \quad \dot\alpha, \dot\beta = 1, 2 . \tag{143}$$

It should be noted at this point that lower undotted indices can be seen as row indices, while upper undotted indices describe columns. The dotted indices follow the opposite convention.

We know that $\eta_{\mu\nu}$ can be used to raise and lower $O(1,3)$ indices and some of you may wonder if the same can be done for Weyl spinors. The answer is yes! We will use the following spinor contractions

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{144}$$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{145}$$

where the dotted indices are just indices for the barred spinor, i.e. $\overline{\psi}$. We can then define

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta} \ , \ \psi_{\alpha} = \epsilon_{\alpha\beta}\psi^{\beta} \ , \tag{146}$$

$$\overline{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\beta}} \ , \ \overline{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\overline{\psi}^{\dot{\beta}} \ . \tag{147}$$

Having seen how Lorentz transformations act on Dirac spinors, we can derive how they act on Weyl spinors. It is easy to check that the matrices that obey the Lorentz algebra and generate our desired transformations are given by[6]

$$(\sigma^{\mu\nu})_{\alpha}^{\ \beta} = -\frac{i}{4}(\sigma^{\mu}\overline{\sigma}^{\nu} - \sigma^{\nu}\overline{\sigma}^{\mu})_{\alpha}^{\ \beta} \tag{148}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} = -\frac{i}{4}(\overline{\sigma}^{\mu}\sigma^{\nu} - \overline{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}_{\ \dot{\beta}} , \tag{149}$$

which tells us that the left and right spinors transform as[7]

$$\psi_{\alpha} \to \left(e^{\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_{\alpha}^{\ \beta}\psi_{\beta} \tag{150}$$

$$\overline{\chi}^{\dot{\alpha}} \to \left(e^{\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)^{\dot{\alpha}}_{\ \dot{\beta}}\overline{\chi}^{\dot{\beta}} \ . \tag{151}$$

4 N=1 super-Poincare Algebra and its Representations

This chapter is based on [2], [4], [6], [7], and [8].

4.1 General Irreducible Representations of the Homogenous Lorentz Group

We know that representations of the Lorentz group must fulfill the Lorentz algebra. In Chapter 2, we used the contravariant notation for our generators. Now, we will use covariant generators to help you differentiate between the generators of the algebra and generators of the representation. We know that the commutation relation is given by

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = i(\mathcal{J}_{\rho\nu}\eta_{\sigma\mu} + \mathcal{J}_{\mu\rho}\eta_{\nu\sigma} - \mathcal{J}_{\sigma\nu}\eta_{\rho\mu} - \mathcal{J}_{\mu\sigma}\eta_{\nu\rho}) \ . \tag{152}$$

Similar to Chapter 2, we will now define two vectors of generators, where

$$\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3) = (\mathcal{J}_{23}, \mathcal{J}_{31}, \mathcal{J}_{12}) \tag{153}$$

$$\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3) = (\mathcal{J}_{10}, \mathcal{J}_{20}, \mathcal{J}_{30}) \ . \tag{154}$$

We then can write the commutation relations as

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k \tag{155}$$

$$[\mathcal{J}_i, \mathcal{K}_j] = i\epsilon_{ijk}\mathcal{K}_k \tag{156}$$

$$[\mathcal{K}_i, \mathcal{K}_j] = -i\epsilon_{ijk}\mathcal{J}_k \ . \tag{157}$$

However, it's very convenient to replace the $\mathcal{J}$ and $\mathcal{K}$ matrices with two decoupled spin three-vectors

$$\mathbf{A} \equiv \frac{1}{2}(\mathcal{J} + i\mathcal{K}) \tag{158}$$

$$\mathbf{B} \equiv \frac{1}{2}(\mathcal{J} - i\mathcal{K}) \ . \tag{159}$$

Our "new" commutation relations are then

$$[A_i, A_j] = i\epsilon_{ijk}A_k \tag{160}$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k \tag{161}$$

$$[A_i, B_j] = 0 \ . \tag{162}$$

Matrices that fulfill this Lie algebra are found in the same way that one searches for matrices representing spins of two uncoupled particles–as a direct sum. This means that we label the rows and columns of these matrices with a pair of integers and/or half-integers a and b, that can take the values

$$a = -A, -A+1, ..., A-1, A \tag{163}$$
$$b = -B, -B+1, ..., B-1, B \tag{164}$$

and use

$$(\mathbf{A})_{a'b',ab} = \delta_{b'b}\mathbf{J}^{(A)}_{a'a} \tag{165}$$
$$(\mathbf{B})_{a'b',ab} = \delta_{b'b}\mathbf{J}^{(B)}_{a'a} , \tag{166}$$

where $\mathbf{J}^{(A)}$ and $\mathbf{J}^{(B)}$ are the spin matrices we have seen in quantum mechanics. The indices A and B denote their respective spin.

$$(\mathbf{J}^{(A)}_3) = a\delta_{a'a} \tag{167}$$
$$(\mathbf{J}^{(A)}_1 \pm i\mathbf{J}^{(A)}_2)_{a'a} = \delta_{a',a\pm1}\sqrt{(A \mp a)(A \pm a + 1)} \tag{168}$$

The same goes for $\mathbf{J}^{(B)}$. Our represenations will be labelled by the values of two positive integers and/or half-integers A and B. This means that the (A, B) representation has dimensionality $(2A + 1)(2B + 1)$ [4].
Those who were paying close attention might have realized something important that I have neglected to mention. Since we started off with Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, this means that $\mathcal{J}$ is Hermitian, while $\mathcal{K}$ is very much anti-Hermitian. This, in turn, tells us something very important about finite-dimensional representations of the Lorentz group. They are NOT unitary. This is not a problem, as we are working with fields, not wave functions. As result, we do not need a Lorentz invariant positive norm.
It should be noted, that the generators of the rotation group are given by the matrices

$$\mathcal{J} = \mathbf{A} + \mathbf{B} . \tag{169}$$

Using the rules of vector addition, it can easily be shown that a field, which lives in the (A, B) representation, has components that rotate as if the had spin j, with

$$j = A + B, ..., |A - B| . \tag{170}$$

We can use this to identify the (A, B) representations with more familiar objects. For example, a $(0, 0)$ field is a scalar with $j = 0$, whereas $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ only allows $j = \frac{1}{2}$. These are our Weyl spinors. The $(\frac{1}{2}, \frac{1}{2})$ field has components with $j = 1$ and $j = 0$, which respectively correspond to the spatial part $\mathbf{v}$ and time-component v^0 of a four-vector v^μ. In general, an (A, A) field contains terms with only integer spins $2A, 2A-1, ..., 1, 0$ and corresponds to a traceless symmetric tensor of rank $2A$. A general tensor of rank N transforms as the direct product of N $(\frac{1}{2}, \frac{1}{2})$ four-vector representations. This can then be decomposed into irreducible terms (A, B), with $A = \frac{N}{2}, \frac{N}{2} - 1, ...$ and $B = \frac{N}{2}, \frac{N}{2} - 1, ...$[4]. It should be noted, that there is a tensor product rule $(\frac{1}{2}, 0)^{\otimes r} \otimes (0, \frac{1}{2})^{\otimes s} = (\frac{s}{2}, \frac{s}{2})$, which just comes from the tensor product rules of $SU(2)$ [7]. Using what we have learned here, we can identify the Poincare generators P^μ and $M^{\mu\nu}$ as (bosonic) generators with $(A, B) = (\frac{1}{2}, \frac{1}{2})$ and $(1, 0) \oplus (0, 1)$ respectively [8].

4.2 Graded Lie Algebra

Definition:
A graded Lie Algebra of grade n is a vector space

$$L = \bigoplus_{i=0}^{i=n} L_i , \tag{171}$$

such that all L_i are vector spaces and the product

$$[\ ,\ \} : L \times L \to L \tag{172}$$

has three properties:

$$[L_i, L_j\} \in L_{i+j}\ mod\ n + 1 \tag{173}$$

$$[L_i, L_j\} = -(-1)^{ij}[L_j, L_i\} \tag{174}$$

$$[L_i, [L_j, L_k\}\}(-1)^{ik} + [L_j, [L_k, L_i\}\}(-1)^{ij} + [L_k, [L_i, L_k\}\}(-1)^{jk} = 0 . \tag{175}$$

The first property, for example, tells us that L_0 is a Lie algebra, while the others are not. For our purpose, which is to create the super-poincare algebra, $n = 1$. L_0 will be the Poincare algebra, whereas $L_1 = (Q^I_\alpha, \overline{Q}^I_\alpha)$, with $I = 1, ..., N$. Q^I_α and $\overline{Q}^I_\alpha$ will be a set of $N + N = 2N$ anticommuting fermionic generators that transform like Weyl spinors.
In our case, the product $[L_i, L_j\}$ will be the commutator if i and/or $j = 0$ and otherwise it will be the anti-commutator $\{\ ,\ \}$, which is defined as

$$\{A, B\} = AB + BA \ . \tag{176}$$

I have used the terms bosonic and fermionic generators once or twice during this paper. To clarify, bosonic operators are elements of L_0, whereas fermionic operators lie in L_1. [6]
It should be noted that since the Q's generate space-time symmetries, we expect them to be representations of the Lorentz group. I mentioned that they will be in the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representation, but how do we see this? We know that if Q^I_α lies in the (j, j'), $\overline{Q}^I_\alpha$ must lie in (j', j) . As a result, the anti-commutator $\{Q^I_\alpha, \overline{Q}^I_\alpha\}$ contains $(j + j', j + j')$, which has to lie in L_0 due to (173). This has to be P^μ, which is $(\frac{1}{2}, \frac{1}{2})$. As a result, Q is either $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ [2].

4.3 N=1 Super-Poincare Algebra

I will now first write down the graded algebra and then set out to prove it. Since we are looking at simple supersymmetry, we expect $I = 1$, which simplifies our set of fermionic generators. Since I is therefore a fixed number, rather then an index, will not explicitly write it.

$$\begin{aligned}
i\,[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} && (177)\\
i\,[P^\mu, J^{\rho\sigma}] &= \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho && (178)\\
[P^\mu, P^\rho] &= 0 && (179)\\
[Q_\alpha, J^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta && (180)\\
[Q_\alpha, P^\mu] &= 0 && (181)\\
\left\{Q_\alpha, Q^\beta\right\} &= 0 && (182)\\
\left\{Q_\alpha, \overline{Q}_{\dot\beta}\right\} &= 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu && (183)
\end{aligned}$$

The first three equations are just the Poincare algebra and therefore don't hold any new information. To show the other four (anti-) commutation relations, we will often implicitly use representation theory to determine what the left-hand side can be equal to.
<u>Claim 13:</u> $[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta$
<u>Proof:</u>
We know that Q_α transforms as a spinor, which tells us that under a infinitessimal Lorentz transformation, we get

$$\begin{aligned}
Q'_\alpha &= \left(e^{\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha^{\ \beta} Q_\beta && (184)\\
&\approx \left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \ . && (185)
\end{aligned}$$

But we also know that Q_α is an operator, which then also must transform as

$$\begin{aligned}
Q'_\alpha &= U^\dagger Q_\alpha U && (186)\\
U &= \left(e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\right) \ . && (187)
\end{aligned}$$

(186) tells us that

$$Q'_\alpha \approx \left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) Q_\alpha \left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) \ . \tag{188}$$

By setting (185) and (188) equal to one another, we get

$$Q_\alpha + \frac{i}{2}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta = Q_\alpha + \frac{i}{2}\omega_{\mu\nu}\left(Q_\alpha J^{\mu\nu} - J^{\mu\nu}Q_\alpha\right) + O(\omega^2) \ , \tag{189}$$

which finally tells us that

$$[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \ . \tag{190}$$

By doing a similar calculation, we also get the corresponding commutation relation of the right-handed representation

$$\left[\overline{Q}^{\dot\alpha}, J^{\mu\nu}\right] = (\overline{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\overline{Q}^{\dot\beta} \ . \tag{191}$$

Claim 14: $[Q_\alpha, P^\mu] = 0$
Proof:
Representation theory, as well as the need for a correct index structure, would suggest that the actual result of our commutator could look like this:

$$[Q_\alpha, P^\mu] = c(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{Q}^{\dot{\alpha}} \ . \tag{192}$$

We must now show that $c = 0$. (192) actually gives us two equations, since we can take the corresponding equation for $\overline{Q}$.

$$[\overline{Q}^{\dot{\alpha}}, P^\mu] = c^\star(\overline{\sigma})^{\dot{\alpha}\alpha}Q_\alpha \ , \tag{193}$$

which can be derived by taking the Hermitian conjugate of (192). As a last step, we will use the Jacobi identity for $P\mu$, P^ν, and Q_α.

$$\begin{aligned}
0 &= [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] && (194)\\
&= -c\,(\sigma^\nu)_{\alpha\dot{\alpha}}\left[P^\mu, \overline{Q}^{\dot{\alpha}}\right] + c\,(\sigma^\mu)_{\alpha\dot{\alpha}}\left[P^\nu, \overline{Q}^{\dot{\alpha}}\right] && (195)\\
&= |c|^2\,(\sigma^\nu)_{\alpha\dot{\alpha}}\,(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}\,Q_\beta - |c|^2\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,(\overline{\sigma}^\nu)^{\dot{\alpha}\beta}\,Q_\beta && (196)\\
&= |c|^2(\sigma^{\nu\mu})_\alpha^{\ \beta}Q_\beta && (197)
\end{aligned}$$

We thusly conclude that $c = 0$.
Claim 15: $\{Q_\alpha, Q^\beta\} = 0$
Proof:
We again use a similar argument and write $\{Q_\alpha, Q^\beta\}$ in a general form

$$\{Q_\alpha, Q^\beta\} = k(\sigma^{\nu\mu})_\alpha^{\ \beta}J_{\mu\nu} \ . \tag{198}$$

We might have expected a $(\sigma^{\mu\nu})P_\mu P_\nu$ term as well. But due to the antisymmetry of $(\sigma^{\mu\nu})$ in μ and ν, no such term could exist. We know that due to Claim 14, the left-hand side manifestly commutes with P. Our Poincare algebra tells us that the right-hand side does not, except when $k = 0$.
Claim 16: $\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu$
Proof:
Once again, we use the same index argument to write this commutation relation as

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = t(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu \ . \tag{199}$$

This time, however, we cannot find an argument to set $t = 0$. By convention, we set $t = 2$.
Let me quickly explain our last relation in words. If $|F\rangle$ represents a fermionic state and $|B\rangle$ a bosonic state, our SUSY algebra tells us that

$$Q|F\rangle = |B\rangle \qquad \overline{Q}|F\rangle = |B\rangle \tag{200}$$
$$Q|B\rangle = |F\rangle \qquad \overline{Q}|B\rangle = |F\rangle \ . \tag{201}$$

Our last anticommutation relation tells us that

$$Q\overline{Q}|B\rangle \sim P|B\rangle \ , \tag{202}$$

which means that the product of these two generators preserves the spin, but translates the particle in spacetime. Thus, the SUSY generators "know" all about spacetime. The Q's are therefore spacetime symmetries, rather than internal symmetries [8].

4.4 Commutators with Internal Symmetries

We know that internal symmetry generators commute with all of the Poincare generators. This carries over to the SUSY algebra. For an internal symmetry generator T_a, we get

$$[T_a, Q_\alpha] = 0 \ . \tag{203}$$

There is one exception though. The SUSY generators create an additional internal symmetry, which is called R-Symmetry. There exists an automorphism that acts on the SUSY algebra and preserves the algebra.

$$\begin{aligned}
Q_\alpha &\to e^{it}Q_\alpha && (204)\\
\overline{Q}_{\dot{\alpha}} &\to e^{-it}\overline{Q}_{\dot{\alpha}} \ , && (205)
\end{aligned}$$

where t is a real parameter. This is a $U(1)$ internal symmetry. Let's call R the generator of the $U(1)$ [8]. Our operators then transform as

$$Q_\alpha \to e^{-iRt}Q_\alpha e^{iRt} \ . \tag{206}$$

By comparing (204)-(206), we get

$$\begin{aligned}
[Q_\alpha, R] &= Q_\alpha && (207)\\
[\overline{Q}_{\dot{\alpha}}, R] &= -\overline{Q}_{\dot{\alpha}} \ . && (208)
\end{aligned}$$

4.5 Representations of the Poincare Group

Before we talk about the representations of the super-Poincare algebra, we need to cover representations of the Poincare group. Our first step will be to find the Casimir operators of the Poincare group. These will be two operators that commute with all of the Poincare generators. To do this, we need to define the Pauli Ljubanski vector W_μ

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma \ , \tag{209}$$

where $\epsilon_{0123} = -\epsilon^{0123} = +1$.
Claim 17: $[W_\mu, P_\nu] = 0$
Proof:

$$\begin{aligned}
[W_\mu, P_\alpha] &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left[J^{\nu\rho}P^\sigma, P_\alpha\right] = -\frac{1}{2}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\left[J^{\nu\rho}P^\sigma, P^\beta\right] && (210)\\
&= -\frac{1}{2}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\left(J^{\nu\rho}\left[P^\sigma, P^\beta\right] + \left[J^{\nu\rho}, P^\beta\right]P^\sigma\right) && (211)\\
&= -\frac{1}{2}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\left(\left[J^{\nu\rho}, P^\beta\right]P^\sigma\right) && (212)\\
&= -\frac{i}{2}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\left(P^\rho\eta^{\beta\nu} - P^\nu\eta^{\beta\rho}\right)P^\sigma && (213)\\
&= \frac{i}{2}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\left(P^\nu\eta^{\beta\rho} - P^\rho\eta^{\beta\nu}\right)P^\sigma && (214)\\
&= \frac{i}{2}\eta_\alpha^{\ \rho}\epsilon_{\mu\nu\rho\sigma}P^\nu P^\sigma - \frac{i}{2}\eta_\alpha^{\ \nu}\epsilon_{\mu\nu\rho\sigma}P^\rho P^\sigma && (215)\\
&= \frac{i}{2}\epsilon_{\mu\nu\alpha\sigma}P^\nu P^\sigma - \frac{i}{2}\epsilon_{\mu\alpha\rho\sigma}P^\rho P^\sigma && (216)\\
&= \frac{i}{2}\epsilon_{\mu\nu\alpha\sigma}P^\nu P^\sigma + \frac{i}{2}\epsilon_{\mu\nu\alpha\sigma}P^\nu P^\sigma && (217)\\
&= i\epsilon_{\mu\nu\alpha\sigma}P^\nu P^\sigma && (218)\\
&= 0 && (219)
\end{aligned}$$

since ϵ is anti-symmetric, whereas $P^\nu P^\rho$ symmetric.
Claim 18: $[W_\mu, J_{\rho\sigma}] = i\eta_{\mu\rho}W_\sigma - i\eta_{\mu\sigma}W_\rho$
Proof:

$$\begin{aligned}
[W_\mu, J_{\rho\sigma}] &= -\frac{1}{2}\epsilon_{\mu\lambda\chi\theta}\left[J^{\lambda\chi}P^\theta, J_{\rho\sigma}\right]\\
&= -\frac{1}{2}\epsilon_{\mu\lambda\chi\theta}\left(J^{\lambda\chi}\left[P^\theta, J_{\rho\sigma}\right] + \left[J^{\lambda\chi}, J_{\rho\sigma}\right]P^\theta\right)\\
&= \frac{1}{2}\epsilon_{\mu\lambda\chi\theta}\left(J^{\lambda\chi}\left(i\eta_\rho^{\ \theta}P_\sigma - i\eta_\sigma^{\ \theta}P_\rho\right) - i\left(J^\lambda_{\ \sigma}\eta_\rho^{\ \chi} + J^\chi_{\ \rho}\eta_\sigma^{\ \lambda} - J^\lambda_{\ \rho}\eta_\sigma^{\ \chi} - J^\chi_{\ \sigma}\eta_\rho^{\ \lambda}\right)P^\theta\right)\\
&= \frac{i}{2}\epsilon_{\mu\lambda\chi\theta}\left(J^{\lambda\chi}\left(\eta_\rho^{\ \theta}P_\sigma - \eta_\sigma^{\ \theta}P_\rho\right) - \left(2J^\chi_{\ \rho}\eta_\sigma^{\ \lambda} - 2J^\lambda_{\ \rho}\eta_\sigma^{\ \chi}\right)P^\theta\right)\\
&= \frac{i}{2}\epsilon_{\mu\lambda\chi\theta}\left(\eta_{\sigma\tau}\eta_\rho^{\ \theta} - \eta_{\rho\tau}\eta_\sigma^{\ \theta}\right)\left(J^{\lambda\chi}P^\tau - 2J^{\lambda\tau}P^\chi\right)
\end{aligned}$$

At this point we will need the identity $\epsilon^{\lambda\chi\tau\gamma}W_\gamma = \left(J^{\lambda\chi}P^\tau - 2J^{\lambda\tau}P^\chi\right)$. This identity will not be proven. We can now turn this into

$$\begin{aligned}
&= \frac{i}{2}\epsilon_{\mu\lambda\chi\theta}\left(\eta_{\sigma\tau}\eta_\rho^{\ \theta} - \eta_{\rho\tau}\eta_\sigma^{\ \theta}\right)\epsilon^{\lambda\chi\tau\gamma}W_\gamma && (220)\\
&= -i\left(\eta_\mu^{\ \tau}\eta_\theta^{\ \gamma} - \eta_\mu^{\ \gamma}\eta_\theta^{\ \tau}\right)\left(\eta_{\sigma\tau}\eta_\rho^{\ \theta} - \eta_{\rho\tau}\eta_\sigma^{\ \theta}\right)W_\gamma && (221)\\
&= i\eta_{\mu\rho}W_\sigma - i\eta_{\mu\sigma}W_\rho \ . && (222)
\end{aligned}$$

Claim 19: $[W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$
Proof:

$$\begin{aligned}
[W_\mu, W_\nu] &= -\frac{1}{2}\epsilon_{\nu\rho\sigma\tau}\left[W_\mu, J^{\rho\sigma}P^\tau\right] && (223)\\
&= -\frac{1}{2}\epsilon_{\nu\rho\sigma\tau}\left[W_\mu, J^{\rho\sigma}\right]P^\tau && (224)\\
&= -\frac{i}{2}\epsilon_{\nu\rho\sigma\tau}\left(\eta_\mu^{\ \rho}W^\sigma - \eta_\mu^{\ \sigma}W^\rho\right)P^\tau && (225)\\
&= -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma && (226)
\end{aligned}$$

Claim 20: $[W^\mu, Q_\alpha] = i\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta P_\nu$
Proof:

$$\begin{aligned}
[W^\mu, Q_\alpha] &= -\frac{1}{2}\eta^{\mu\nu}\epsilon_{\nu\rho\sigma\tau}\left[J^{\rho\sigma}P^\tau, Q_\alpha\right] && (227)\\
&= \frac{1}{2}\epsilon^\mu{}_{\rho\sigma\tau}\left(\sigma^{\rho\sigma}\right)_\alpha^{\ \beta} Q_\beta P^\tau && (228)\\
&= i\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta P_\nu && (229)
\end{aligned}$$

The Poincare Casimirs are given by

$$\begin{aligned}
C_1 &= P^\mu P_\mu && (230)\\
C_2 &= W^\mu W_\mu \ . && (231)
\end{aligned}$$

We can use Claims 17-20 to show that they indeed commute with all Poincare generators, but not with the Qs.
Poincare multiplets are then labelled by $|m, \omega\rangle$, where m and w are the quantum numbers associated with C_1 and C_2 respectively. The eigenvalue p^μ of the generator P^μ will be used as a label for states within the representation. To find more labels, we will start with P^μ as given and look for additional quantum numbers.
Massive Particles
For massive particles we can choose the rest frame, where $p^\mu = (m, 0, 0, 0)$. For W_μ, we then get

$$\begin{aligned}
W_0 &= 0 && (232)\\
W_i &= -mJ_i \ , && (233)
\end{aligned}$$

where the J_i satisfy the rotation group algebra. Every irreducible representation is given by $|m, j, p^\mu, j_3\rangle$ and describes a massive particle of spin j.
Massless Particles
A massless particle's momentum is given by $p^\mu = (E, 0, 0, E)$, which tells us that

$$\begin{aligned}
&(W_0, W_1, W_2, W_3) = E\left(J_3, -J_1 + K_2, -J_2 - K_1, -J_3\right) && (234)\\
\implies\ &[W_1, W_2] = 0\ ,\ [W_3, W_1] = -iEW_2\ ,\ [W_3, W_2] = iEW_1\ . && (235)
\end{aligned}$$

These are the commutation of the Euclidean group in two dimensions. For finite dimensional representations W_1, as well as W_2, have to be zero. This tells us that $W^\mu = \lambda P^\mu$. We can then label our states $|0, 0, p^\mu, \lambda\rangle =: |p^\mu, \lambda\rangle$, where we call λ the helicity. Under CPT, the helicity changes sign, which tells us that our state $|p^\mu, \lambda\rangle$ is mapped to $|p^\mu, -\lambda\rangle$. The relation

$$exp\left(2\pi i\lambda\right)|p^\mu, \lambda\rangle = \pm|p^\mu, \lambda\rangle \tag{236}$$

tells us that λ must be either an integer or a half-integer [7].

4.6 N=1 Supersymmetry Representations

For simple supersymmetry, C_1 is still a good Casimir, whereas C_2 is not. This tells us that one can have particles of different spin within our supermultiplet. The new Casimir operator, $\widetilde{C}_2$, is defined by

$$\begin{aligned}
B_\mu &:= W_\mu - \frac{1}{4}\overline{Q}_{\dot\alpha}\left(\overline{\sigma}_\mu\right)^{\dot\alpha\beta} Q_\beta && (237)\\
C_{\mu\nu} &:= B_\mu P_\nu - P_\nu B_\mu && (238)\\
\widetilde{C}_2 &:= C_{\mu\nu}C^{\mu\nu} \ . && (239)
\end{aligned}$$

Claim 21: The number n_B of bosons equals the number n_F of fermions in every supermultiplet, i.e.

$$n_B = n_F \ . \tag{240}$$

Proof:
We first create the fermion number operator $(-1)^F = (-)^F$, which is defined as

$$\begin{aligned}
(-)^F|B\rangle &= |B\rangle && (241)\\
(-)^F|F\rangle &= -|F\rangle \ . && (242)
\end{aligned}$$

We have now created something that anti-commutes with Q_α, since

$$\begin{aligned}
(-)^F Q_\alpha|F\rangle = (-)^F|B\rangle &= |B\rangle = Q_\alpha|F\rangle = -Q_\alpha(-)^F|F\rangle && (243)\\
\implies \left\{(-)^F, Q_\alpha\right\} &= 0 \ . && (244)
\end{aligned}$$

To attain our desired result, we must first calculate the following trace:

$$Tr\left\{(-)^F\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\}\right\} = Tr\left\{(-)^F Q_\alpha \overline{Q}_{\dot{\beta}} + (-)^F \overline{Q}_{\dot{\beta}} Q_\alpha\right\} \tag{245}$$

$$= Tr\left\{-Q_\alpha (-)^F \overline{Q}_{\dot{\beta}} + Q_\alpha (-)^F \overline{Q}_{\dot{\beta}}\right\} \tag{246}$$

$$= 0 \,, \tag{247}$$

where we used (244) and the fact that the trace is invariant under cyclic permutation.
We also know that $\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$, which tells us that

$$Tr\left\{(-)^F\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\}\right\} = Tr\left\{(-)^F 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu\right\} \tag{248}$$

$$= 2(\sigma^\mu)_{\alpha\dot{\beta}} p_\mu Tr\left\{(-)^F\right\} \,. \tag{249}$$

We thusly get

$$0 = Tr\left\{(-)^F\right\} = \sum_{bosons} \langle B|(-)^F|B\rangle + \sum_{fermions} \langle F|(-)^F|F\rangle \tag{250}$$

$$= \sum_{bosons} \langle B|B\rangle - \sum_{fermions} \langle F|F\rangle \tag{251}$$

$$= n_B - n_F \,. \tag{252}$$

4.7 Massless Supermultiplet

As in section 4.5, the P_μ-eigenvalues for massless particles can be written as $p_\mu = (E, 0, 0, E)$. In this case both Casimirs will be zero. We know that

$$\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu = 2E\left(\sigma^0 + \sigma^3\right)_{\alpha\dot{\beta}} \tag{253}$$

$$= 4E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \,, \tag{254}$$

which proves that Q_2 is zero:

$$\left\{Q_2, \overline{Q}_{\dot{2}}\right\} = 0 \implies \langle p^\mu, \lambda|\overline{Q}_{\dot{2}} Q_2|\tilde{p}^\mu, \tilde{\lambda}\rangle = 0 \tag{255}$$

$$\implies Q_2 = 0 \tag{256}$$

The equation for Q_1, on the other hand, can be written as $\left\{Q_1, \overline{Q}_{\dot{1}}\right\} = 4E$. We now define creation- and annihilation operators a and $a^\dagger$

$$a := \frac{Q_1}{2\sqrt{E}} \tag{257}$$

$$a^\dagger := \frac{\overline{Q}_{\dot{1}}}{2\sqrt{E}} \,, \tag{258}$$

which then give us the anti-commutation relations

$$\left\{a, a^\dagger\right\} = 1 \tag{259}$$

$$\{a, a\} = \left\{a^\dagger, a^\dagger\right\} = 0 \,. \tag{260}$$

We also know that because $\left[a, J^3\right] = \frac{1}{2}\left(\sigma^3\right)_{11} a = \frac{1}{2} a$

$$J^3\left(a|p^\mu, \lambda\rangle\right) = \left(aJ^3 - \left[a, J^3\right]\right)|p^\mu, \lambda\rangle \tag{261}$$

$$= \left(aJ^3 - \frac{a}{2}\right)|p^\mu, \lambda\rangle \tag{262}$$

$$= \left(\lambda - \frac{1}{2}\right) a|p^\mu, \lambda\rangle \,, \tag{263}$$

which tells us that $a|p^\mu, \lambda\rangle$ has helicity $\lambda - \frac{1}{2}$. Doing a similar calculation, we find that $a^\dagger|p^\mu, \lambda\rangle$ has helicity $\lambda + \frac{1}{2}$. To build a representation, we will start with the lowest helicity state, which we will call the vacuum state $|\Omega\rangle$. It should be clear that $a|\Omega\rangle = 0$ and $a^\dagger a^\dagger|\Omega\rangle = 0|\Omega\rangle = 0$ [7]. The whole multiplet therefore consists of

$$|\Omega\rangle = |p^\mu, \lambda\rangle \tag{264}$$

$$a^\dagger|\Omega\rangle = |p^\mu, \lambda + \frac{1}{2}\rangle \,. \tag{265}$$

Since we need CPT invariance, we must add the CPT conjugate to attain the final result

$$|p^\mu, \pm\lambda\rangle \ , \ |p^\mu, \pm(\lambda + \frac{1}{2})\rangle \ . \tag{266}$$

4.8 Massive Supermultiplet

For $m \neq 0$, we get P^μ-eigenvalues $p^\mu = (m, 0, 0, 0)$ and Casimirs

$$\begin{aligned} C_1 &= P^\mu P_\mu = m^2 &(267)\\ \widetilde{C}_2 &= C_{\mu\nu}C^{\mu\nu} = 2m^4 Y^i Y_i \ , &(268) \end{aligned}$$

where Y_i is called the superspin.

$$\begin{aligned} Y_i &= J_i - \frac{1}{4m}\overline{Q}\overline{\sigma}_i Q = \frac{B_i}{m} &(269)\\ [Y_i, Y_j] &= i\epsilon_{ijk}Y_k &(270) \end{aligned}$$

The eigenvalues of $Y^2 = Y^i Y_i$ are $y(y+1)$, which allows us to label the irreducible representations by $|m, y\rangle$. Once again, we will use the anti-commutation relation for Q and $\overline{Q}$ to get the states.

$$\begin{aligned} \left\{Q_\alpha, \overline{Q}_{\dot\beta}\right\} &= 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu = 2m\left(\sigma^0\right)_{\alpha\dot\beta} &(271)\\ &= 2m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\beta} &(272) \end{aligned}$$

We now have two non-zero anti-commutation relations, which allows us to define two sets of ladder operators

$$\begin{aligned} a_{1,2} &:= \frac{Q_{1,2}}{\sqrt{2m}} &(273)\\ a^\dagger_{1,2} &:= \frac{\overline{Q}_{\dot1,\dot2}}{\sqrt{2m}} \ , &(274) \end{aligned}$$

which then have the following anti-commutation relations

$$\begin{aligned} \left\{a_p, a^\dagger_q\right\} &= \delta_{pq} &(275)\\ \{a_p, a_q\} = \left\{a^\dagger_p, a^\dagger_q\right\} &= 0 \ . &(276) \end{aligned}$$

Let's again define a vacuum state $|\Omega\rangle$, which is annihilated by $a_{1,2}$. We therefore get

$$\begin{aligned} Y_i|\Omega\rangle &= J_i|\Omega\rangle - \frac{1}{4m}\overline{Q}\overline{\sigma}_i\sqrt{2m}a|\Omega\rangle &(277)\\ &= J_i|\Omega\rangle \ . &(278) \end{aligned}$$

This means that the spin number j and superspin number y are the same for $|\Omega\rangle$. So, for a given m, y,

$$|\Omega\rangle = |m, j = y, p^\mu, j_3\rangle \ . \tag{279}$$

We obtain the rest using

$$\begin{aligned} a_1|j_3\rangle &= |j_3 - \frac{1}{2}\rangle \ , \ a^\dagger_1|j_3\rangle = |j_3 + \frac{1}{2}\rangle &(280)\\ a_2|j_3\rangle &= |j_3 + \frac{1}{2}\rangle \ , \ a^\dagger_2|j_3\rangle = |j_3 - \frac{1}{2}\rangle \ , &(281) \end{aligned}$$

which tells us that when the $a^\dagger_p$ act on $|\Omega\rangle$, they behave like a coupling of two spins j and $\frac{1}{2}$. As a result, this yields a linear combination of two spins $j + \frac{1}{2}$ and $j - \frac{1}{2}$ with Clebsch Gordan coefficients k_i

$$\begin{aligned} a^\dagger_1|\Omega\rangle &= k_1|m, j = y + \frac{1}{2}, p^\mu, j_3 + \frac{1}{2}\rangle + k_2|m, j = y - \frac{1}{2}, p^\mu, j_3 + \frac{1}{2}\rangle &(282)\\ a^\dagger_2|\Omega\rangle &= k_3|m, j = y + \frac{1}{2}, p^\mu, j_3 - \frac{1}{2}\rangle + k_4|m, j = y - \frac{1}{2}, p^\mu, j_3 - \frac{1}{2}\rangle \ . &(283) \end{aligned}$$

The fourth and final state is then

$$a^\dagger_2 a^\dagger_1|\Omega\rangle = -a^\dagger_1 a^\dagger_2|\Omega\rangle \propto |\Omega\rangle \ , \tag{284}$$

which represents a spin j object. In total, we have

$$
\begin{aligned}
2 \quad &\cdot \quad |m, j = y, p^\mu, j_3\rangle && (285)\\
1 \quad &\cdot \quad |m, j = y + \frac{1}{2}, p^\mu, j_3\rangle && (286)\\
1 \quad &\cdot \quad |m, j = y - \frac{1}{2}, p^\mu, j_3\rangle \ . && (287)
\end{aligned}
$$

We will now discuss the case $y = 0$ separately.

$$
\begin{aligned}
|\Omega\rangle \quad &= \quad |m, j = 0, p^\mu, j_3 = 0\rangle && (288)\\
a^\dagger_{1,2}|\Omega\rangle \quad &= \quad |m, j = \frac{1}{2}, p^\mu, j_3 = \pm\frac{1}{2}\rangle && (289)\\
a^\dagger_1 a^\dagger_2|\Omega\rangle \quad &= \quad |m, j = 0, p^\mu, j_3 = 0\rangle =: |\Omega'\rangle && (290)
\end{aligned}
$$

Since J is parity invariant, whereas K picks up a minus sign, parity interchanges (A, B) and (B, A). So, for example, $(\frac{1}{2}, 0) \leftrightarrow (0, \frac{1}{2})$. Let us now consider the two $j = 0$ states $|\Omega\rangle$ and $|\Omega'\rangle$, where the first is annihilated by a_i and the second one by $a^\dagger_i$. Because parity interchanges Q and $\overline{Q}$, parity interchanges a_i and $a^\dagger_i$ and therefore $|\Omega\rangle \leftrightarrow |\Omega'\rangle$ [7]. To attain two states with defined parity, we need linear combinations

$$
\begin{aligned}
|\pm\rangle \quad &:= \quad |\Omega\rangle \pm |\Omega'\rangle && (291)\\
P|\pm\rangle \quad &= \quad \pm|\pm\rangle \ . && (292)
\end{aligned}
$$

5 Wess-Zumino Model

This chapter is based on [2] and [9]. Before we continue our talk about the general supersymmetrtic algebra and its representations, I will introduce one of the simplest supersymmetric models in four spacetime dimensions. In section 4.8, we analyzed to case for $y = 0$ and got a supermultiplet with a single Majorana fermion and two complex bosonic fields, which can be separated into a complex scalar and pseudoscalar field. We can then split our complex fields into two real fields. So, what we get in total are one Majorana field ψ, one real scalar and pseudoscalar bosonic field, which we will call A and B, respectively,and last but not least, a real scalar and pseudoscalar bosonic auxiliary field F and G [9]. Before I introduce the model in its non-interacting and interacting form, I will introduce two new matrices and the index manipulations that come with them.

5.1 γ^5 and γ_5

In chapter 3, I introduced four matrices, γ^i, where $i = 0, 1, 2, 3$. These could be written as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \ , \tag{293}$$

where

$$\sigma^\mu = \left(\mathbb{1}, \sigma^i\right) \text{ and } \overline{\sigma}^\mu = \left(\mathbb{1}, -\sigma^\mathrm{i}\right) \ . \tag{294}$$

We can then define γ^5 as

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{295}$$

In addition, we can also try to lower the indices of our gamma matrices [2]. We do this by lowering the indices of our Pauli matrices which then gives us

$$\sigma_\mu = \left(\mathbb{1}, -\sigma^i\right) \text{ and } \overline{\sigma}_\mu = \left(\mathbb{1}, \sigma^\mathrm{i}\right) \ . \tag{296}$$

Our "new" gamma matrices are then

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix} \tag{297}$$

and γ_5 is then given by

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \ . \tag{298}$$

5.2 On-Shell Non-Interacting Model

Without further ado, here is the free model Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\left(\partial_\mu A\right)\left(\partial^\mu A\right) - \frac{1}{2}m^2A^2 + \frac{1}{2}\left(\partial_\mu B\right)\left(\partial^\mu B\right) - \frac{1}{2}m^2B^2 + \frac{i}{2}\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}m\overline{\psi}\psi \\ &= L_0 + L_m \ . \end{aligned}$$

By taking the Euler-Lagrange equations, we get that

$$\begin{aligned} \left(\Box + m^2\right) A &= 0 && (299) \\ \left(\Box + m^2\right) B &= 0 && (300) \\ \left(i\partial\!\!\!/ - m\right)\psi &= 0 \ , && (301) \end{aligned}$$

where $\Box$ is the d'Alembert operator. This tells us that our scalar bosons satisfy the Klein-Gordon equation, whereas our fermion fulfills the Dirac equation [9].

5.3 Off-Shell Interacting Model

In addition to terms that will depend on the mass, some will now have a coupling constant g. Our Lagrangian is then given by

$$\begin{aligned} L = &\ \frac{1}{2}\left(\partial_\mu A\right)\left(\partial^\mu A\right) + \frac{1}{2}\left(\partial_\mu B\right)\left(\partial^\mu B\right) + \frac{i}{2}\overline{\psi}\partial\!\!\!/\psi + \frac{1}{2}\left(F^2 + G^2\right) && (302) \\ &+m\left(FA + GB - \frac{1}{2}\overline{\psi}\psi\right) && (303) \\ &+g\left(F\left(A^2 - B^2\right) + 2GAB - \overline{\psi}\left(A - i\gamma_5 B\right)\psi\right) && (304) \\ = &\ L_0 + L_m + L_g \ , && (305) \end{aligned}$$

which then give us the following Euler-Lagrange equations

$$\begin{aligned} 0 &= -\Box A + mF + 2gFA + 2gGB - g\overline{\psi}\psi && (306) \\ 0 &= -\Box B + mG - 2gFB + 2gGA + ig\overline{\psi}\gamma_5\psi && (307) \\ 0 &= F + mA + 2g\left(A^2 - B^2\right) && (308) \\ 0 &= B + mB + 2gAB && (309) \\ 0 &= i\partial\!\!\!/\psi - m\psi - 2g\left(A - i\gamma_5 B\right)\psi \ . && (310) \end{aligned}$$

We see that two of the equations do not include derivatives. This allows us to rewrite F and G purely in terms of A and B. We now see why they are called auxiliary fields. They are not actually needed to write our Lagrangian. If we rewrite our Lagrangian without F and G, we then get the on-shell formalism [9].

5.4 On-Shell Interacting Model

Whereas the off-shell Lagrangian didn't really resemble the free model, our new on-shell Lagrangian contains three extra terms, but will otherwise look identical to it [9]. By plugging in (308) and (309) into the off-shell Lagrangian, we get

$$\begin{aligned} L &= \frac{1}{2}\left(\partial_\mu A\right)\left(\partial^\mu A\right) + \frac{1}{2}\left(\partial_\mu B\right)\left(\partial^\mu B\right) - \frac{1}{2}m^2\left(A^2 + B^2\right) + \frac{i}{2}\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}m\overline{\psi}\psi \\ &\quad -mgA\left(A^2 + B^2\right) - \frac{1}{2}g^2\left(A^2 + B^2\right)^2 - g\overline{\psi}\left(A - i\gamma_5 B\right)\psi \ , \end{aligned}$$

which in turn gives us the following three EL-Equations.

$$\begin{aligned} \left(\Box + m^2\right) A &= -mg\left(3A^2 + B^2\right) - 2g^2A\left(A^2 + B^2\right) - g\overline{\psi}\psi && (311) \\ \left(\Box + m^2\right) B &= -2mgAB - 2g^2B\left(A^2 + B^2\right) + ig\overline{\psi}\gamma_5\psi && (312) \\ \left(i\partial\!\!\!/ - m\right)\psi &= 2g\left(A - i\gamma_5 B\right)\psi \ . && (313) \end{aligned}$$

5.5 Conserved Supercurrent

Noether's theorem states that each continuous symmetry leads to a conserved current. We know that the Wess-Zumino model acts on a massive supermultiplet, which tells us that each of the SUSY generators should leave the Lagrangiang invariant (up to a divergence). We will now look at only one of them and show that there is indeed a conserved supercurrent. Notice that our result should not depend on whether we work with the off-shell or on-shell model. In our case, using the terms F and G will help our steps look more compact, which means that we will use off-shell notation from now on. We will now show that the following supersymmetric transformation indeed creates a conserved quantity.

$$\delta_\epsilon A = \bar{\epsilon}\psi \tag{314}$$

$$\delta_\epsilon B = i\bar{\epsilon}\gamma_5\psi \tag{315}$$

$$\delta_\epsilon \psi = -i\partial\!\!\!/ \left(A + i\gamma_5 B\right)\epsilon + \left(F + i\gamma_5 G\right)\epsilon \tag{316}$$

$$\delta_\epsilon F = -i\bar{\epsilon}\partial\!\!\!/\psi \tag{317}$$

$$\delta_\epsilon G = \bar{\epsilon}\gamma_5\partial\!\!\!/\psi \ , \tag{318}$$

where ϵ is a constant infinitessimal Majorana field. This obviously transforms fermions into bosons and vice versa, which should reaffirm us that we are indeed working with a supersymmetric transformation. The conjugate spinor, $\overline{\psi}$, is then varied by

$$\delta_\epsilon\overline{\psi} = \delta_\epsilon\psi^\dagger\gamma^0 = i\bar{\epsilon}\partial\!\!\!/\left(A - i\gamma_5 B\right) + \bar{\epsilon}\left(F + i\gamma_5 G\right) \ . \tag{319}$$

To prove the invariance of the Lagrangian, we will see how L_0, L_m, and L_g transform. For L_0, we first need that

$$\begin{aligned}
\frac{1}{2}\delta_\epsilon\left(\partial^\mu A\partial_\mu A + \partial^\mu B\partial_\mu B\right) &= \bar{\epsilon}\partial^\mu\left(A + i\gamma_5 B\right)\partial_\mu\psi \\
\frac{1}{2}\delta_\epsilon\left(F^2 + G^2\right) &= -i\bar{\epsilon}\left(F + i\gamma_5 G\right)\partial\!\!\!/\psi \\
\delta_\epsilon\left(\overline{\psi}\partial\!\!\!/\psi\right) &= i\bar{\epsilon}\partial\!\!\!/\left(A - i\gamma_5 B\right)\partial\!\!\!/\psi + \bar{\epsilon}\left(F + i\gamma_5 G\right)\partial\!\!\!/\psi \\
&\quad -i\bar{\epsilon}\Box\left(A + i\gamma_5 B\right)\psi - \bar{\epsilon}\partial\!\!\!/\left(F - i\gamma_5 G\right)\psi \ ,
\end{aligned}$$

which are derived with the help of the following identities

$$\bar{\epsilon}\psi = \overline{\psi}\epsilon \tag{320}$$

$$\bar{\epsilon}\gamma_5\psi = \overline{\psi}\gamma_5\epsilon \tag{321}$$

$$\bar{\epsilon}\gamma^\mu\psi = -\overline{\psi}\gamma^\mu\epsilon \tag{322}$$

$$\bar{\epsilon}\gamma_5\gamma^\mu\psi = \overline{\psi}\gamma_5\gamma^\mu\epsilon \ . \tag{323}$$

This then tells us that the variation of L_0 is given by

$$\delta_\epsilon L_0 = \partial_\mu\left(\bar{\epsilon}V_0^\mu\right) \tag{324}$$

$$V_0^\mu = \frac{1}{2}\gamma^\mu\left\{\partial\!\!\!/\left(A + i\gamma_5 B\right) - i\left(F - i\gamma_5 G\right)\right\}\psi \ . \tag{325}$$

We have now shown that the massless, non-interacting term is invariant under this susy transformation. Our next step will be to show that the mass term is invariant as well. For this, we'll need that

$$\delta_\epsilon\left(FA + GB\right) = \bar{\epsilon}\left(F + i\gamma_5 G\right)\psi - i\bar{\epsilon}\left(A + i\gamma_5 B\right)\partial\!\!\!/\psi \tag{326}$$

$$\delta_\epsilon\left(\overline{\psi}\psi\right) = 2\bar{\epsilon}\left(F + i\gamma_5 G\right)\psi + 2i\bar{\epsilon}\partial\!\!\!/\left(A - i\gamma_5 B\right)\psi \ , \tag{327}$$

which in turn tells us that

$$\delta_\epsilon L_m = \partial_m\left(\bar{\epsilon}V_m^\mu\right) \tag{328}$$

$$V_m^\mu = -im\left(A + i\gamma_5 B\right)\gamma^\mu\psi \ . \tag{329}$$

Before we do the same for our interaction term, we first need that

$$\begin{aligned}
\delta_\epsilon\left(FA^2 - FB^2\right) &= \bar{\epsilon}\left\{2F\left(A - i\gamma_5 B\right) - i\left(A^2 - B^2\right)\partial\!\!\!/\right\}\psi \\
\delta_\epsilon\left(2ABG\right) &= 2\bar{\epsilon}\left(AB\gamma_5\partial\!\!\!/ + BG + iAG\gamma_5\right)\psi \\
\delta_\epsilon\left(A\overline{\psi}\psi\right) &= \left(\bar{\epsilon}\psi\right)\left(\overline{\psi}\psi\right) + 2A\bar{\epsilon}\left\{F + i\gamma_5 G + i\partial\!\!\!/\left(A - i\gamma_5 B\right)\right\}\psi \\
\delta_\epsilon\left(\overline{\psi}\gamma_5\psi\right) &= 2\bar{\epsilon}\left\{\left(F + i\gamma_5 G\right) + i\partial\!\!\!/\left(A - \gamma_5 B\right)\right\}\gamma_5\psi \\
\delta_\epsilon\left(B\overline{\psi}\gamma_5\psi\right) &= i\left(\bar{\epsilon}\gamma_5\psi\right)\left(\overline{\psi}\gamma_5\psi\right) + 2B\bar{\epsilon}\left\{\left(F + i\gamma_5 G\right) + i\partial\!\!\!/\left(A - i\gamma_5 B\right)\right\}\gamma_5\psi \ .
\end{aligned}$$

Equipped with these equations, as well as the fact that $\left(\overline{\psi}\psi\right)\left(\overline{\epsilon}\psi\right)+\left(\overline{\psi}\gamma_5\psi\right)\left(\overline{\epsilon}\gamma_5\psi\right)=0$ (This will not be proven, but for those who are interested, this can be derived from the Fiertz identities for Majorana spinors), we can find out that the variation of L_g is

$$\begin{aligned} \delta_\epsilon L_g &= \partial_\mu\left(\overline{\epsilon}V_g^\mu\right) &(330)\\ V_g^\mu &= -ig\left(A+i\gamma_5 B\right)^2\gamma^\mu\psi\ . &(331)\end{aligned}$$

We have finally shown that each of the three components are invariant under this transformation, which tells us that our Lagrangian is invariant and

$$\begin{aligned} \delta_\epsilon L &= \partial_\mu\left(\overline{\epsilon}V^\mu\right)\\ V^\mu &= \gamma^\mu\left\{\frac{1}{2}\partial\!\!\!/\left(A+i\gamma_5 B\right)-\frac{i}{2}\left(F-i\gamma_5 G\right)-im\left(A-i\gamma_5 B\right)-ig\left(A-i\gamma_5 B\right)^2\right\}\psi\ .\end{aligned}$$

Our Noether current is then given by

$$J^\mu=\frac{\partial L}{\partial\left(\partial_\mu\phi\right)}\delta_\epsilon\phi-V^\mu\ , \tag{332}$$

where ϕ will run over all our fields. We have our V^μ, which means that we only need to determine

$$\frac{\partial L}{\partial\left(\partial_\mu\phi\right)}\delta_\epsilon\phi\ . \tag{333}$$

By using another Majorana identity, namely $\overline{\psi}\partial\!\!\!/\psi=\partial_\mu\overline{\psi}\gamma^\mu\psi$, we get that

$$\begin{aligned}\frac{\partial L}{\partial\left(\partial_\mu\phi\right)}\delta_\epsilon\phi &= \partial^\mu A\overline{\epsilon}\psi+i\partial^\mu B\overline{\epsilon}\gamma_5\psi-\frac{i}{2}\delta_\epsilon\overline{\psi}\gamma^\mu\psi\\ &= \overline{\epsilon}\left\{2\partial^\mu\left(A+i\gamma_5 B\right)-\frac{1}{2}\gamma^\mu\partial\!\!\!/\left(A+i\gamma_5 B\right)-\frac{i}{2}\left(F-i\gamma_5 G\right)\right\}\psi\ .\end{aligned}$$

Having finally calulated everything we need [9], we get that

$$\begin{aligned}\overline{\epsilon}J^\mu &= \overline{\epsilon}\left\{\partial\!\!\!/\left(A-i\gamma_5 B\right)\gamma^\mu\psi+im\gamma^\mu\left(A-i\gamma_5 B\right)\psi+ig\gamma^\mu\left(A-i\gamma_5 B\right)^2\right\}\psi\\ &= -i\delta_\epsilon\overline{\psi}\gamma^\mu\psi=i\overline{\psi}\gamma^\mu\delta_\epsilon\psi\ .\end{aligned}$$

6 Extended Supersymmetry

This chapter is based on [6], [7], and [10].

6.1 Extended Supersymmetry Algebra

Having previously discussed simple symmetry in chapter 4, as well as analyzed one potential model, we will now turn to the more general case of $N>1$, which is known as extended supersymmetry. Our Q-generators now obtain an additional label $A,B=1,...,N$ (It does not matter whether this index is on top or at the bottom. We are not somehow contracting over it) and our extended algebra stays the same with two exceptions.

$$\begin{aligned}\left\{Q_\alpha^A,\overline{Q}_{\dot{\beta}B}\right\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta^A_B &(334)\\ \left\{Q_\alpha^A,Q_\beta^B\right\} &= \epsilon_{\alpha\beta}Z^{AB}\ , &(335)\end{aligned}$$

where the Z^{AB} are in the $(0,0)$ representation (Internal Symmetries) and are known as the central charges [7]. The second equation is derived in the same way we attained most of the results in chapter 4. We know that the $(0,0)$ generators are the only possible elements that can be on the right-hand-side and the $\epsilon_{\alpha\beta}$ gives us the correct index structure. All possible additional factors can be "pulled into" the central charges. Since $\epsilon_{\alpha\beta}$ is anti-symmetric, we can choose our central charges to be anti-symmetric as well. In addition, the Z^{AB} commute with everything. We will now try to motivate equation (334).
Claim 22: $\left\{Q_\alpha^A,\overline{Q}_{\dot{\beta}B}\right\}=2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta^A_B$
Proof:
We would in general expect our expression to be of the form $\left\{Q_\alpha^A,\overline{Q}_{\dot{\beta}B}\right\}=2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu C^A_B$, where C^A_B is some matrix. By

taking the hermitian conjugate, we get that $C_B^A = C_A^{B*}$, which tells us that C is in fact hermitian. There is therefore a unitary transformation U, which diagonalizes C. Our Qs then transform as

$$Q_\alpha^A \rightarrow U_K^A Q_\alpha^K \tag{336}$$
$$\overline{Q}_{\dot\beta}^B \rightarrow \overline{Q}_{\dot\beta}^L (U^{-1})_L^{\ B} \ . \tag{337}$$

We have now found the supercharges that give us a diagonal matrix $C = diag(c_l)$, but we would now like this matrix to be δ_B^A. To obtain this result, we will transform our Qs anew. Since all diagonal elements must be positive (energies must be positive)

$$Q_\alpha^A \rightarrow \sqrt{c_l} Q_\alpha^A \tag{338}$$
$$\overline{Q}_{\dot\beta}^B \rightarrow \sqrt{c_l}\overline{Q}_{\dot\beta}^B \ . \tag{339}$$

We have now found a basis of supercharges that fulfill equation (334) [10].

6.2 Massless Representations for N > 1

This section will be very similar to 4.7. We start with $p_\mu = (E, 0, 0, E)$, which then gives us

$$\left\{Q_\alpha^A, \overline{Q}_{\dot\beta B}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta_B^A = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta_B^A \tag{340}$$
$$\Longrightarrow \quad Q_2^{\ A} = 0 \ . \tag{341}$$

Equation (305) then immediately tells us that our central charges must vanish. Once again, we will use our Qs as ladder operators,

$$a^A := \frac{Q_1^{\ A}}{2\sqrt{E}} \tag{342}$$
$$a^{A\dagger} := \frac{\overline{Q}_{\dot 1}^{\ A}}{2\sqrt{E}} \tag{343}$$
$$\Longrightarrow \quad \left\{a^A, a^{A\dagger}\right\} = \delta_B^A \ . \tag{344}$$

We start with a vacuum state $|\Omega\rangle$, which is annihilated by the a^A.

states	helicity	number of states
$\lvert\Omega\rangle$	λ_0	$1 = \binom{N}{0}$
$a^{A\dagger}\lvert\Omega\rangle$	$\lambda_0 + \frac{1}{2}$	$N = \binom{N}{1}$
$a^{A\dagger}a^{B\dagger}\lvert\Omega\rangle$	$\lambda_0 + 1$	$\frac{1}{2!}N(N-1) = \binom{N}{2}$
.	.	.
.	.	.
.	.	.
$a^{N\dagger}a^{(N-1)\dagger}...a^{1\dagger}\lvert\Omega\rangle$	$\lambda_0 + \frac{N}{2}$	$1 = \binom{N}{N}$

This gives us a total number of 2^N states, since

$$\sum_{k=0}^{N} \binom{N}{k} = \sum_{k=0}^{N} \binom{N}{k} 1^k 1^{N-k} = 2^N \ , \tag{345}$$

where we used the binomial theorem in the last step. Before I conclude this section, there are a couple of interesting results I'd like to discuss. First of all, the maximal difference of helicities in a supermulitplet is $\lambda_{max} - \lambda_{min} = \frac{N}{2}$. We can use this fact to find the maximal value for N. There is a strong belief amongst physicists that there are no massless particles with helicity $|\lambda| > 2$. This, in turn, tells us that N can be no greater than 8 [7].

6.3 Massive Representations for N > 1

We will start out with $p_\mu = (m, 0, 0, 0)$, which tells us that

$$\left\{Q_\alpha^A, \overline{Q}_{\dot\beta B}\right\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\beta} \delta_B^A \ . \tag{346}$$

In contrast to the previous chapter, our anti-commutation relations do not require our central charges to vanish. We will therefore look at two cases: one where $Z^{AB}=0$ and one where $Z^{AB}\neq 0$.
Case 1: $Z^{AB}=0$
We have $2N$ pairs of creation and annihilation operators

$$a_\alpha^A := \frac{Q_\alpha^A}{\sqrt{2m}} \tag{347}$$

$$a_{\dot\alpha}^{A\dagger} := \frac{\overline{Q}_{\dot\alpha}^A}{\sqrt{2m}}\ , \tag{348}$$

which leads to 2^{2n} particles. Each of these particles has a spin j (not necessarily the same j for each particle), where j_3 can go from $-j$ to j [7]. Let us look at one example, where $N=2$ and the vacuum state $|\Omega\rangle$ has spin 0. We then get

$$\begin{array}{cc}
|\Omega\rangle & 1\times\text{spin }0\\
a_{\dot\alpha}^{A\dagger}|\Omega\rangle & 4\times\text{spin }\frac{1}{2}\\
a_{\dot\alpha}^{A\dagger}a_{\dot\beta}^{B\dagger}|\Omega\rangle & 3\times\text{spin }0\ ,\ 3\times\text{spin }1\\
a_{\dot\alpha}^{A\dagger}a_{\dot\beta}^{B\dagger}a_{\dot\gamma}^{C\dagger}|\Omega\rangle & 4\times\text{spin }\frac{1}{2}\\
a_{\dot\alpha}^{A\dagger}a_{\dot\beta}^{B\dagger}a_{\dot\gamma}^{C\dagger}a_{\dot\delta}^{D\dagger}|\Omega\rangle & 1\times\text{spin }0
\end{array}$$

Case 2: $Z^{AB}\neq 0$
Here, there are two cases; one where N is even and one where N is odd. Without proving it, I will now rely on a linear algebra theorem, which states that if N is even, we can find a basis, such that our central charge matrix can be written as

$$Z^{AB} = \begin{pmatrix}
0 & Z_1 & & & & & & \\
-Z_1 & 0 & & & & & & \\
 & & 0 & Z_2 & & & & \\
 & & -Z_2 & 0 & & & & \\
 & & & & ... & ... & & \\
 & & & & ... & ... & & \\
 & & & & & & 0 & Z_{\frac{N}{2}} \\
 & & & & & & Z_{\frac{N}{2}} & 0
\end{pmatrix} . \tag{349}$$

If, however, our N is odd, we can find a matrix with the same structure as before, only with an additional column and row that consist of only zeros. So without loss of generality, we will assume N is even and continue our analysis. In this basis, we define our annihilation operators as

$$a_\alpha^1 = \frac{1}{\sqrt{2}}\left(Q_\alpha^1+\epsilon_{\alpha\beta}\left(Q_\beta^2\right)^\dagger\right) \tag{350}$$

$$b_\alpha^1 = \frac{1}{\sqrt{2}}\left(Q_\alpha^1-\epsilon_{\alpha\beta}\left(Q_\beta^2\right)^\dagger\right) \tag{351}$$

$$a_\alpha^2 = \frac{1}{\sqrt{2}}\left(Q_\alpha^3+\epsilon_{\alpha\beta}\left(Q_\beta^4\right)^\dagger\right) \tag{352}$$

$$b_\alpha^2 = \frac{1}{\sqrt{2}}\left(Q_\alpha^3-\epsilon_{\alpha\beta}\left(Q_\beta^4\right)^\dagger\right) \tag{353}$$

$$... = ... \tag{354}$$

$$... = ... \tag{355}$$

$$a_\alpha^{\frac{N}{2}} = \frac{1}{\sqrt{2}}\left(Q_\alpha^{N-1}+\epsilon_{\alpha\beta}\left(Q_\beta^N\right)^\dagger\right) \tag{356}$$

$$b_\alpha^{\frac{N}{2}} = \frac{1}{\sqrt{2}}\left(Q_\alpha^{N-1}-\epsilon_{\alpha\beta}\left(Q_\beta^N\right)^\dagger\right)\ . \tag{357}$$

Our creation operators are defined accordingly. We then get the following anti-commutation relations between our ladder operators

$$\left\{a_\alpha^r,\left(a_\beta^s\right)^\dagger\right\} = (2m+Z_r)\,\delta_{rs}\delta_{\alpha\beta} \tag{358}$$

$$\left\{b_\alpha^r,\left(b_\beta^s\right)^\dagger\right\} = (2m-Z_r)\,\delta_{rs}\delta_{\alpha\beta} \tag{359}$$

$$\left\{a_\alpha^r,\left(b_\beta^s\right)^\dagger\right\} = \left\{a_\alpha^r,a_\beta^s\right\} = ... = 0\ . \tag{360}$$

Since we expect non-negative values on the right-hand-side, we get that

$$2m \geq |Z_r| \text{ for r} = 1, 2, .., \frac{\text{N}}{2} \ . \tag{361}$$

Now, for $2m > |Z_r|$, we get 2^{2N} states, where 2^{2N-1} are bosonic and 2^{2N-1} are fermionic. In addition, if our vacuum state has spin λ_0, our states will have spins that vary from $\lambda_0 - \frac{N}{2}$ to $\lambda_0 + \frac{N}{2}$.
But what happens when $2m = |Z_r|$ for some r? One of the anticommutations vanishes and that particular set of operators becomes trivial. This mirrors the situation we had with massless particles, where half of the operators "vanished". Our multiplet shortens and the dimension of our representation is reduced accordingly [6]. If this "trivialization" occurs k times, we get the following representations

$$k = 0 \quad \Longrightarrow \quad 2^{2N} \ particles, \ "long \ mulitplet" \tag{362}$$

$$0 < k < \frac{N}{2} \quad \Longrightarrow \quad 2^{2(N-k)} \ particles, \ "short \ mulitplet" \tag{363}$$

$$k = \frac{N}{2} \quad \Longrightarrow \quad 2^{N} \ particles, \ "ultra - short \ mulitplet" \ . \tag{364}$$

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