Proseminar FS18 - Algebra, Topology and Group Theory in Physics

The Dirac monopole and the 't Hooft-Polyakov monopole

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This report is the result of a synthesis of publications on magnetic monopoles, meaning particles behaving like isolated magnetic poles. It was created in the framework of a presentation given for a seminar on algebra, topology and group theory at the ETH Zürich in the spring semester 2018.

Contents

Introduction

A question that one might ask himself when hearing about monopoles is why we study them. Indeed, until now, magnetic monopoles have not been observed experimentally.
We will see here that they appear naturally in gauge theories, which are at the basis of modern physics.
The aim of this report is to give a few insights in how to get monopoles by changing the topological structure of the space we are working with or how they arise in gauge theories and a deep consequence that would result from the existence of magnetic monopoles.

THE DIRAC MONOPOLE

Covariant derivative and wavefunction phase

Let us consider the Hamiltonian of a charged particle of mass m and charge q in a quantum mechanical setting with classical electromagnetic background:

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi \ . \tag{1}$$

The canonical momentum is computed as follows:

$$\frac{dr_i}{dt} = -\frac{i}{\hbar}[r_i, H] = \frac{p_i - qA_i}{2m} \ , \tag{2}$$

$$\Rightarrow \Pi = m\frac{d\vec{r}}{dt} = \vec{p} - q\vec{A} \ , \tag{3}$$

By promoting $\vec{p}$ to an operator $-i\hbar\vec{\nabla}$, we get the following commutation relation:

$$[\Pi_i, \Pi_j] = -i\hbar(\partial_i A_j - \partial_j A_i) = -i\hbar q \epsilon_{ijk} B_k \ , \tag{4}$$

$$[\Pi_i, H] = i\hbar q\Big(-\partial_i\Phi + \epsilon_{ijk}\frac{1}{2m}(\Pi_j B_k - B_j \Pi_k)\Big) \ , \tag{5}$$

$$\Rightarrow m\frac{d^2\vec{r}}{dt^2} = q\Big(-\vec{\nabla}\Phi + \frac{1}{2}(\frac{d\vec{r}}{dt}\wedge\vec{B} - \vec{B}\wedge\frac{d\vec{r}}{dt})\Big) \ . \tag{6}$$

The Schrödinger equation for an arbitrary normalised wavefunction $|\Psi, t\rangle$ will then become:

$$i\hbar\frac{d}{dt}|\Psi, t\rangle = (\frac{\Pi^2}{2m} + q\Phi)|\Psi, t\rangle \ , \tag{7}$$

$$= \Big(-\frac{\hbar^2}{2m}(\vec{\nabla} - \frac{i}{\hbar}\vec{A})\underbrace{(\vec{\nabla} - \frac{i}{\hbar}\vec{A})}_{=:\vec{\nabla}_c} + q\Phi\Big)|\Psi, t\rangle \ . \tag{8}$$

Where $\vec{\nabla}_c$ is the covariant derivative of the theory.
We know that the Maxwell equations are invariant under the "Gauge transformation": $\vec{A} \mapsto \vec{A} - \vec{\nabla}\chi$, therefore, it is to be expected that the physics of the system does not change under such a transformation. Indeed, all our wavefunction does is to pick up a phase:

$$\langle\vec{r}|\Psi, t\rangle \mapsto e^{\frac{i}{\hbar}q\chi(\vec{r})}\langle\vec{r}|\Psi, t\rangle \ . \tag{9}$$

As can be seen by a short computation, the derivative $\nabla\langle\vec{r}|\Psi, t\rangle$ does not transform as the wavefunction does:

$$\nabla e^{\frac{i}{\hbar}q\chi(\vec{r})}\langle\vec{r}|\Psi, t\rangle = \underbrace{q\frac{i}{\hbar}\nabla\chi e^{\frac{i}{\hbar}q\chi(\vec{r})}\langle\vec{r}|\Psi, t\rangle}_{\text{additional term}} + \underbrace{e^{\frac{i}{\hbar}q\chi(\vec{r})}\nabla\langle\vec{r}|\Psi, t\rangle}_{\text{well-transforming term}} \ . \tag{10}$$

On the other hand, the covariant derivative we had above transforms just as we wished it (the vector notation will be dropped, when not needed, we will expect the reader to be able to determine if grad, div or curl is meant):

$$\nabla_c \mapsto \nabla_c - q\frac{i}{\hbar}\nabla\chi =: \nabla_c{}' \ , \tag{11}$$

$$\begin{aligned}\Rightarrow \overbrace{(\nabla - q\frac{i}{\hbar}A - q\frac{i}{\hbar}\nabla\chi)}^{=\nabla_c{}'} e^{\frac{i}{\hbar}q\chi(\vec{r})} \langle\vec{r}|\Psi,t\rangle \\ = e^{\frac{i}{\hbar}q\chi(\vec{r})} \underbrace{(\nabla - q\frac{i}{\hbar}A)}_{=\nabla_c} \langle\vec{r}|\Psi,t\rangle \ .\end{aligned} \tag{12}$$

If our particle moves in a space free of magnetic field, we can still perform a Gauge transformation, we have $\vec{B} = \vec{\nabla} \wedge \vec{A} = 0$, therefore there exists a scalar functions $\chi(\vec{r})$ such that: $\chi = \int_\gamma \vec{A}d\vec{l}$ (where $\gamma(t)$ is the path taken by the particle through space). One should note that the intergal is invariant under homotopic path taken by the particles as long as no electromagnetic source is enclosed in between two such paths, one should recall the Aharonov-Bohm effect. We get that the wavefunction picks up a phase:

$$\langle\vec{r}|\vec{a},t\rangle \mapsto e^{\frac{i}{\hbar}q\int_{\vec{a}}^{\vec{r}}\vec{A}d\vec{l}}\langle\vec{r}|\vec{a},t\rangle_0 \ . \tag{13}$$

This phase will be of crucial importance in the discussion of the Dirac quantization condition given in the third section.

Role of non-trivial topology

Theorem (Stokes' Theorem [1])**.** *Let M be a m-dimensional differentiable manifold.*
Let $\omega \in \Omega^{r-1}(M)$ be a (r-1)-form on M.

$$\int_M d\omega = \int_{\partial M} \omega \tag{14}$$

definition. A differential form $\omega \in \Omega^r(M)$ is called **closed** if $d\omega = 0$ and **exact** if $\exists\alpha \in \Omega^{r-1}$ such that $\omega = d\alpha$.

Theorem (Poincaré's Lemma [1])**.** *If a coordinate neighbourhood U of a manifold M is contractible to a point $p_0 \in M$, any closed r-form on U is also exact.*

The magnetic field $\vec{B}$ can be interpreted as a 2-form and the vector potential as a 1-form. We know that $div(\vec{B}) = \underbrace{dB}_{\text{diff. form}} = 0$, which implies that no magnetic charge density appears (compare to $div(\vec{E}) = \frac{\rho}{\epsilon_0}$), hence, no magnetic monopoles. Lets imagine $dB \neq 0$ on a finite region of space, we then get magnetic charges, if we take that region out, we keep our charges, but get a non trivial topological space with $dB = 0$ everywhere. Next to that, we can, by **Poincaré's Lemma**, locally find vector potentials which will give us the correct magnetic field.

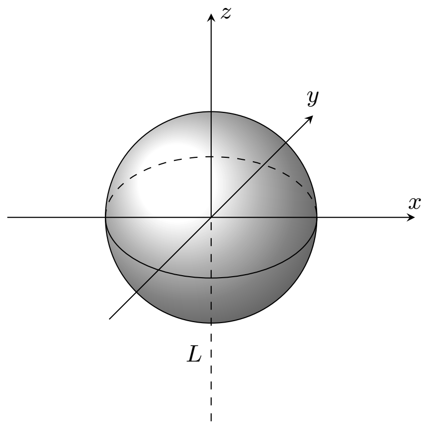

FIG. 1: This represents the patch $U_N = \mathbb{R}^3 \setminus (-\infty,\ 0)$ on which $\vec{A}_N$ has no singularities. The path L (dashed z-axis) is the one with respect to with we will integrate to find $\vec{A}_N$. (see below)

Construction of the Dirac monopole

What we will do now is to make our space topologically non-trivial, i.e. non-contractible. The simplest way to do that is to remove a point, we will remove the origin of our space: $\mathbb{R}^3 \setminus \{0\}$. This space is obviously not contractible anymore. A "physical" argument could be that we first assume the divergence of $\vec{B}$ to be non-zero, so we get a magnetic charge distribution and then "compress" all the charges at the origin. We then get a delta function at the origin and we remove that point from our space (picture a Gaussian repartition converging towards a delta function).
By doing that, our space becomes a manifold that cannot be covered completely by one patch, we will need (at least) two patches:

$$\begin{aligned} U_N &:= \{\vec{x} \in \mathbb{R}^3 \setminus \{0\} |\ 0 \leq \theta \leq \tfrac{\pi}{2} + \delta,\ 0 < r,\ 0 \leq \phi < 2\pi\} \\ U_S &:= \{\vec{x} \in \mathbb{R}^3 \setminus \{0\} |\ \tfrac{\pi}{2} - \delta < \theta < \pi,\ 0 < r,\ 0 \leq \phi < 2\pi\} \\ &\text{with } 0 < \delta < \tfrac{\pi}{2} \end{aligned}$$

Each one of these patches is simply connected, therefore, by the **Poincaré's Lemma** , there exists a vector potential $\vec{A}$, such that $\vec{B} = \vec{\nabla} \wedge \vec{A}$. For the northern patch, a vector potential can be found using the following classical electrodynamics formula [4] , and integrating along the path L as in FIG. 1 as follows:

$$\begin{aligned}
\vec{A}(\vec{x})_N &= g\int_L \frac{d\vec{l'}\wedge(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \\
&\overbrace{=}^{\mathrm{L\equiv L}(t)} \int_{-\infty}^{0} \frac{1}{\sqrt{a+bt+t^2}^3}dt \begin{pmatrix}-y\\x\\0\end{pmatrix} \\
&= \Big[-\frac{2b}{(b^2-4a)\sqrt{a}}-\frac{4}{b^2-4a}\Big]_{-\infty}^{0}\begin{pmatrix}-y\\x\\0\end{pmatrix} \\
&= \dots \overbrace{=}^{\text{spherical coord.}} \frac{g}{r}\frac{(1-\cos(\theta))}{\sin(\theta)}\underbrace{\begin{pmatrix}\sin(\phi)\\-\cos(\phi)\\0\end{pmatrix}}_{=:\vec{e_\phi}} \\
&= \frac{g}{r}\tan(\frac{\theta}{2})\vec{e_\phi}\ . \qquad (15)
\end{aligned}$$

By computing the curl of that vector potential in spherical coordinates, one finds a monopole-like magnetic field:

$$\underbrace{\vec{B}(\vec{r}) = \vec{\nabla}_{\text{sphere}}\wedge\vec{A}(\vec{x}) = g\frac{\vec{r}}{|\vec{r}|^3}}_{\textit{Dirac monopole}}\ . \qquad (16)$$

One can perform a similiar computation to find:

$$\vec{A}_S = -\frac{g}{r}\frac{(1+\cos(\theta))}{\sin(\theta)}\vec{e}_\phi\ . \qquad (17)$$

on the southern patch which will give rise to the same magnetic field, only with the "Dirac-String" (i.e. the divergent part of the vector potentials if considered defined on the whole $\mathbb{R}^3$) in the northern hemisphere, therefore not in the considered patch.
One can see that on the thickened equatorial plane, the two potentials are related by a gauge transformation:

$$\vec{A}_N - \vec{A}_S = \vec{\nabla}\chi\ , \qquad (18)$$

$$\text{with: } \chi = -2g\phi\ . \qquad (19)$$

Therefore, a particle moving into this B-field picks up a phase (13) when going from one patch to another.

One things Dirac noted in his paper from 1931 [5] is that χ does not need to be continuous, only $e^{\frac{i}{\hbar}q\chi(\vec{r})}$ needs to be. We can see that the presence of magnetic charges is directly related to an discontinuity in χ at $0-2\pi$:

$$\begin{aligned}
&\text{let } E = \{\vec{x}\in\mathbb{R}^3|\ |\vec{x}|\le 1\} \\
&\text{let } E_{N/S} = \{\vec{x}\in\mathbb{R}^3|\ |\vec{x}|\le 1,\ x_3 \gtrless 0\} \\
4\pi\ g &= \int_{E(0)} dB \\
&= \int_{E_N} dB + \int_{E_S} dB \\
&\overbrace{=}^{(14)} \int_{\partial B_N} B + \int_{\partial B_S} B \\
&= \int_{U_N\cap S^2} B + \int_{U_S\cap S^2} B \\
&= \int_{U_N\cap S^2} dA_N + \int_{U_S\cap S^2} dA_S \qquad (20) \\
&\overbrace{=}^{(14)} \int_{\partial(U_N\cap S^2)} A_N - \int_{\partial(U_S\cap S^2)} A_S \\
&= \int_{S^1_{equator}} A_N - A_S \\
&\text{"="} \int_{S^1_{equator}} d\chi \\
&\overbrace{=}^{(14)} \chi(2\pi)-\chi(0)\ .
\end{aligned}$$

(The last two steps are not mathematically rigorous, because if χ would be defined that way, it would not be a single-valued function. It is still useful if compared to the properties of χ in the section above.)

Our requirement that the phase gained by the wave-function is well defined and independent from the gauge chosen implies that:

$$\begin{aligned}
e^{\frac{i}{\hbar}q\oint A_N} &= e^{\frac{i}{\hbar}q\oint A_S}\ , \\
\Rightarrow\quad & \frac{1}{\hbar}q4\pi g = 2\pi\ , \\
\Rightarrow\quad & \underbrace{qg = \frac{\hbar}{2}\ n \text{ with } n\in\mathbb{Z}}_{\textit{Dirac quantization condition}}\ . \qquad (21)
\end{aligned}$$

Which means that the existence of magnetic charges implies the well-known quantization of the electric charge, which is observed experimentally!

T'HOOFT MONOPOLE AND BPS-BOUND

SU(2)-Yang-Mills-Higgs Field equations

Let's choose $G = SU(2)$ as gauge group. We shall now derive the field equations for such a theory.
Let $\mathcal{A}_\mu \in su(2)\otimes\Omega^1(\mathbb{R}^4)$ and $\Phi\in su(2)\otimes\Omega^0(\mathbb{R}^4)$ be the gauge potential and the Higgs field. The covariant derivative of the Higgs field and the Yang-Mills field stenght

are:

$$D_\mu\Phi = \partial_\mu\Phi + [\mathcal{A}_\mu, \Phi] \ , \quad (22)$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \ . \quad (23)$$

We choose a basis $\{T_\alpha = i\tau_\alpha | \alpha = 1,2,3\}$ for $su(2)$, where we take the Pauli matrices τ_α, with commutation relation: $[T_\alpha, T_\beta] = -2\epsilon^{\alpha\beta\gamma}T_\gamma$ and normalisation $Tr(T_\alpha T_\beta) = -2\delta_{\alpha\beta}$.
As Lorentz invariant Lagrangian density, we take:

$$\mathcal{L} = \frac{1}{8}Tr(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) - \frac{1}{4}Tr(D_\mu\Phi D^\mu\Phi) - \frac{\lambda}{4}(1+\frac{1}{2}Tr((\Phi)^2))^2 \quad (24)$$

The Lagrangian will the be given by $L = \int d^3x\mathcal{L}$ and the action becomes $\mathcal{S} = \int dtL = \int dtd^3x\mathcal{L} = \int d^4x\mathcal{L}$.

By using the variational principle, we get the equations of motions for our fields:

$$D_\mu D^\mu\Phi = \lambda(1 - Tr((\Phi)^2))\Phi \ , \quad (25)$$

$$D_\mu\mathcal{F}^{\mu\nu} = [D^\nu\Phi, \Phi] \ . \quad (26)$$

These equations are non-linear PDE's of which no general solution is known.

From the Lagrangian above, we can also extract a kinetic and potential term (the kinetic term will vanish when static fields are considered):

$$\begin{aligned} L &= T - V \ , \\ T &= \int d^3x(-\frac{1}{4}Tr(E_iE_i) - \frac{1}{4}Tr(D_0\Phi D_0\Phi)) \ , \quad (27) \\ V &= \int d^3x(-\frac{1}{8}Tr(F_{ij}F_{ij}) - \frac{1}{4}Tr(D_i\Phi D_i\Phi) \quad (28) \\ &\quad + \frac{\lambda}{4}(1 + \frac{1}{2}Tr((\Phi)^2))^2) \ . \end{aligned}$$

Here $E_i = F_{0i}$ and $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$ correspond to the $SU(2)$ electric and magnetic fields.

The vacuum manifold of our theory is defined as: $\mathcal{V} = \{\Phi, \ \mathcal{A} | \ \frac{\lambda}{4}(1 + \frac{1}{2}Tr((\Phi)^2))^2 = 0, \ D_i\Phi = 0$ and $F_{ij} = 0\}$, which represents the field configuration such that the potential energy takes its minimal value. Φ has non-zero expectation value, this breaks the symmetry of $SU(2)$ down to $U(1)$. The unbroken symmetry corresponds to gauge transformation $g(x^\mu) \in SU(2)$ such that $g(x^\mu)T_3g^{-1}(x^\mu) = T_3$. Such symmetries preserve Φ and change $\mathcal{A}_i$, but keep the conditions $D_i\Phi = 0$ and $F_{ij} = 0$. Therefore our vacuum manifold is invariant under $U(1)$ gauge transformations.
Special aspects of symmetry breaking won't be discussed any further here.

Linearised field equations

If one imposes $\Phi \to T_3$ for $r \to \infty$, then the vacuum manifold $\mathcal{V}$ is characterised by $\pi_2(U(1)) = \pi_2(S^1) = 0$ and all fields are trivial (no topological charge).
One boundary condition that can be imposed without restricting us too much is the following: $\Phi(0,0,x_3) \to T_3$ for $x_3 \to \infty$.
We therefore get gauge transformations such that: $g(0,0,x_3) \to Id$ for $x_3 \to \infty$. (for more information one can take a look at the report of Hanno Bertle on symmetry breaking)

Our only hope to find somewhat well-behaved equations is to linearise the above.
Set $\Phi = (1+\phi)T_3$ and $\mathcal{A}_\mu = W^1_\mu T_1 + W^2_\mu T_2 + a_\mu T_3$, with ϕ, W^1_μ, W^2_μ and a_μ small.

We then get the following linear PDE's:

$$\partial_\mu\partial^\mu\phi = -2\lambda\phi \ , \quad (29)$$

$$\partial_\mu(\partial^\mu W^{1\nu} - \partial^\nu W^{1\mu}) = -4W^{1\nu} \ , \quad (30)$$

$$\partial_\mu(\partial^\mu W^{2\nu} - \partial^\nu W^{2\mu}) = -4W^{2\nu} \ , \quad (31)$$

$$\partial_\mu(\partial^\mu a^\nu - \partial^\nu a^\mu) = 0 \ . \quad (32)$$

Let's discuss the static fields. The equations of motions of our fields are the following:

$$D_iD_i\Phi = -\lambda(1 + \frac{1}{2}Tr(\Phi^2))\Phi \ , \quad (33)$$

$$D_iF_{ij} = -[D_j\Phi, \Phi] \ . \quad (34)$$

To solve this, we can use the spherically, rotationally symmetric ansatz:

$$\Phi = h(r)\frac{x^a}{r}T_a \ , \quad (35)$$

$$\mathcal{A}_i = -\frac{1}{2}(1 - k(r))\epsilon^{ija}\frac{x^j}{r^2}T_a \ . \quad (36)$$

where h(r) and k(r) are differentialbe functions of the distance from the origin.

We then get the following ODE's:

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2}{r^2}k^2h - \lambda(1 - h^2)h \ , \quad (37)$$

$$\frac{d^2k}{dr^2} = \frac{1}{r^2}(k^2 - 1)k + 4h^2k \ . \quad (38)$$

To have a singularity free solution at the origin, we impose the boundary conditions: $h(0) = 0$, $k(0) = 1$ and $\lim_{r\to\infty} h(r) = 1$, $\lim_{r\to\infty} k(r) = 0$. This meant that the potential Φ will asymptotically lie in our vacuum manifold.

Numerical Solution

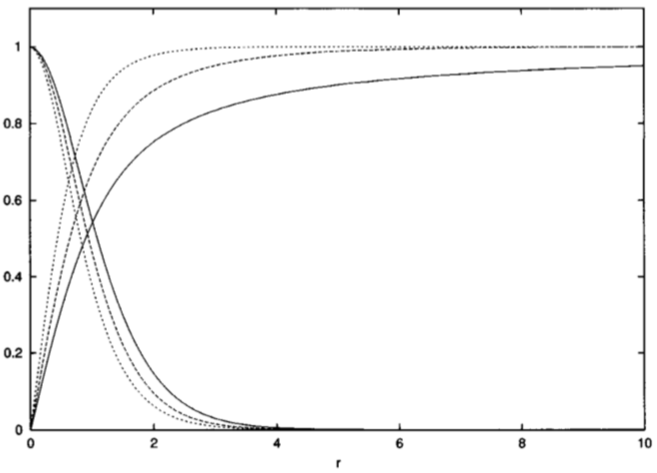

FIG. 2: The monopole profile functions $h(r)$ and $k(r)$ for $\lambda = 0$ (solid curves) (analytical solution), $\lambda = 0.1$ (dashed curves), and $\lambda = 1.0$ (dotted curves)

Magnetic field in SU(2) theory

If we assume that the Higgs field in some region of space can be written as: $\Phi = h\hat{\Phi}$, with $h > 0$ and $\frac{1}{2}Tr(\hat{\Phi}^2) = -1$. We also suppose that $D_\mu\hat{\Phi} = 0$. This condition can be solves by the following general solution, the gauge potential:

$$\mathcal{A}_\mu = \frac{1}{4}[\partial_\mu\hat{\Phi}, \Phi] + a_\mu\hat{\Phi} \ , \tag{39}$$

where a_μ is an arbitrary smooth vector field. The field tensor becomes:

$$F_{\mu\nu} = (\frac{1}{8}Tr([\partial_\mu\hat{\Phi}, \partial_\nu\hat{\Phi}]\hat{\Phi}) + \partial_\mu a_\nu - \partial_\nu a_\mu)\hat{\Phi} \ . \tag{40}$$

which points in $\hat{\Phi}$ direction in $su(2)$.

From the conditions above, the covariant derivative of the Φ is: $D_\mu\Phi = (\partial_\mu h)\hat{\Phi}$.

Therefore, we can define the "Maxwell field tensor" of our theory:

$$\begin{aligned} f_{\mu\nu} &= -\frac{1}{2}Tr(F_{\mu\nu}\hat{\Phi}) \qquad (41)\\ &= \frac{1}{8}Tr([\partial_\mu\hat{\Phi}, \partial_\nu\hat{\Phi}]\hat{\Phi}) + \partial_\mu a_\nu - \partial_\nu a_\mu \ . \end{aligned}$$

The Yang-Mills-Higgs equations then become:

$$\partial_\mu\partial^\mu h = \lambda(1 - h^2)h \ , \tag{42}$$
$$\partial_\mu f^{\mu\nu} = 0 \ . \tag{43}$$

For static fields satifying the conditions above, we can interpret

$$\begin{aligned} b_i &= -\frac{1}{2}\epsilon_{ijk}f_{jk} \qquad (44)\\ &= -\frac{1}{2}\epsilon_{ijk}(\frac{1}{8}Tr([\partial_j\hat{\Phi}, \partial_k\hat{\Phi}]\hat{\Phi}) + \partial_j a_k - \partial_k a_j) \ . \end{aligned}$$

which satisfies $\vec{\nabla} \wedge \vec{b} = 0$ and $\vec{\nabla} \cdot \vec{b} = 0$.
For $r \to \infty$, $h(r) \to 1$, which gives us:

$$b_i = -\frac{x_i}{2r^3} \ . \tag{45}$$

which gives a magnetic monopole of charge -2π.

So our field satisfies the Maxwell-equations known from classical electrodynamics, while asymptotically looking like a monopole. But note that the actual field is smooth everywhere in space, compared to the dirac monople, there is no singularity. But it is impossible to correctly define the electromagnetic field and therefore its divergence in the core region of the monopole due to the fact that the full $SU(2)$ theory is acting there.

BPS-Bound

Analytic Solution ($\lambda = 0$)

In the case $\lambda = 0$, the equations simplify to:

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2}{r^2}k^2 h \ , \tag{46}$$
$$\frac{d^2k}{dr^2} = \frac{1}{r^2}(k^2 - 1)k + 4h^2 k \ . \tag{47}$$
$$\tag{48}$$

The solutions found by Prasad and Sommerfield where:

$$h(r) = \coth(2r) - \frac{1}{2r} \ , \tag{49}$$
$$k(r) = \frac{2r}{\sinh(2r)} \ . \tag{50}$$

Bogomolny realised that the energy of a static field can be written as:

$$\begin{aligned} V &= \int d^3x(-\frac{1}{8}Tr(B_iB_i) - \frac{1}{4}Tr(D_i\Phi D_i\Phi)\\ &= -\frac{1}{4}\int d^3x Tr((B_i + D_i\Phi)^2)\\ &\quad + \frac{1}{2}\int d^3x\partial_i(Tr(B_i\Phi)) \ , \qquad (51) \end{aligned}$$

with $B_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk}$, by using Stoke's Theorem, and the definition of our magnetic field b_i in (44), the equation above further transforms to:

$$V = -\frac{1}{4}\int d^3xTr((B_i + D_i\Phi)^2) - \int_{S^2_\infty} b_i dS^i \quad (52)$$

$$= -\frac{1}{4}\int d^3xTr((B_i + D_i\Phi)^2) + 2\pi N \ . \quad (53)$$

with $N \in \mathbb{Z}_{\geq 0}$ which corresponds to the degree of the map $\Phi = \hat{\Phi}$ (at infinity). More about this will be said in the next section.

$$\text{We get } V = \underbrace{E}_{\text{static field}} \geq 2\pi N \ , \quad (54)$$

$$\text{The equality holds if: } \underbrace{B_i = -D_i\Phi}_{\text{Bogomolny equation}} \ . \quad (55)$$

For the spherically symmetric ansatz, we get the coupled differential equations:

$$\frac{dh}{dr} = \frac{1}{2r^2}(1 - k^2) \ , \quad (56)$$

$$\frac{dk}{dr} = -2hk \ . \quad (57)$$

The Bogomolny equation is the condition for a monopole of topological charge N to be a global energy minimum. A minimum solution should be stationary in the smooth function space. This can be verified y the following computation:

$$D_iD_i\Phi = -D_iB_i = 0 \quad (58)$$

For negative N we find analogous bound, therefore the global result is:

$$E \geq 2\pi|N| \ . \quad (59)$$

The Prasad-Sommerfield solution has charge $N = 1$.

RELATION BETWEEN 'T HOOFT AND DIRAC'S MONOPOLE

Lets consider the section *Magnetic field in SU(2) theory*, we have a field that asymptotically lies in the vacuum manifold of our fields.

We can define the magnetic charge as:

$$g = -\int_{S^2_\infty} f \ , \quad (60)$$

$$\text{with: } f = \frac{1}{8}Tr([d\hat{\Phi}, d\hat{\Phi}]\hat{\Phi}) + da \ , \quad (61)$$

By ($\star$), the magnetic charge becomes the following expression:

$$g = -\frac{1}{8}\int_{S^2_\infty} Tr([d\hat{\Phi}, d\hat{\Phi}]\hat{\Phi}) \ . \quad (62)$$

At infinity, the maps $\hat{\Phi}: \ S^2_\infty \to$ Vacuum, it can be checked that using the normalisation conditions for the generator and the vacuum conditions on the Higgs field, one finds that the vacuum corresponds to S^2 in $su(2)$. Indeed, if one uses that the Higgs field can be written as follows:

$\Phi = aT_1 + bT_2 + cT_3$,

and that in $\mathcal{V}$ we have this condition: $-\frac{1}{2}Tr(\Phi^2) = 1$,

and the normalisation $[T_\alpha, T_\beta] = -2\epsilon^{\alpha\beta\gamma}T_\gamma$,

we get: $\{a,\ b,\ c \in \mathbb{R} |\ a^2 + b^2 + c^2 = 1\} \equiv S^2$. (63)

Where a, b and c correspond to the values of the components of the Higgs evaluated at spatial infinity.
This map has a topological invariant $N \in \pi_2(S^2) = \mathbb{Z}$ called its degree (second homotopy class). The right hand side of the equation above is equal to -2π times this degree, therefore the equation further simplifies to:

$$g = -2\pi N \ . \quad (64)$$

This condition looks like the Dirac quantisation condition from the first section. Indeed there is a connection between the two:

In our gauge: $\hat{\Phi} = T_3$ and has degree N. From topology, we know that unless $N = 0$, $\hat{\Phi}$ is not homotopic to a constant map, however if we divide S^2_∞ in U_N and U_S, we have two contractible space in which we can bring $\hat{\Phi}$ to a constant map, by gauge transformation $g^{(N)}$ and $g^{(S)}$. On the equator of S^2_∞, the two potentials overlap are related by the gauge transformation $g^{(N)}{g^{(S)}}^{-1}$. This gauge transformation preserves $\hat{\Phi}$ (by adjoint action) and therefore lies in $U(1) \subset SU(2)$. So $g^{(N)}{g^{(S)}}^{-1} = \exp(\chi(\phi)T_3)$, where ϕ parametrises the position at the equator. Because the gauge transformations are welldefined in their respective regions, so also on the equator and we obtain the condition: $\chi(2\pi) = \chi(0) + 2\pi N$.

Before our Lie-algebra valued 4 vector field a_μ was well defined over S^2_∞, now we have two vector fields well defined over their respective patches and related by a gauge transformation on the overlap:

$$a^{(N)} - a^{(S)} = -d\chi(\phi) \ , \quad (65)$$

Therefore the 2-form field strength tensor is defined as in the Dirac monopole section:

$$f = da^{(N)} = da^{(S)} \ . \quad (66)$$

therefore its flux through is equal to $-2\pi\tilde{N}$. One can show that $\tilde{N} = N$ and we get back the dirac quantisation condition!

One can explicitly show the the gauge transformations above, if define over the whole space, will give rise to a Dirac string due to the fact that they cannot be continuous over the whole space, see [6, chap.4.5] for a computation of these transformations.

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