The Goldstone theorem and Higgs mechanism are two concepts that are essential to understanding complex phenomena such as superconductivity and weak-interactions. First, the concepts of global, local and gauge symmetries are discussed. Thereafter, the process of spontaneous symmetry breaking is investigated leading to the Goldstone theorem and Higgs mechanism, illustrated with examples. Finally, a group theoretical approach is taken, providing a different perspective to the concepts introduced.
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I. INTRODUCTION

Our best description of nature on the (sub)atomic scale consists of field theories, where interactions and physical quantities are viewed as having values at every point in space-time.

The description of fields in terms of quantum mechanics is known as quantum field theory. Together with aspects of continuous symmetry transformations, (quantum) field theory have been very fruitful for our understanding of the physical world.

Field theories describing a physical system are usually gauge theories, i.e. the Lagrangian of the theory is invariant under a group of local transformations. A local transformation depends on the position in space-time, so the transformation acts differently on the system at every point in space-time. In contrast to this, there is also the concept of a global symmetry transformation, which is characterised by the fact that the transformation acts the same throughout space-time.

Quantum field theories can be thought of as arising from the quantisation of classical field theories. Therefore, it is worth looking at classical field theory in more detail, since this will still provide rich insights. One important idea is that of spontaneous symmetry breaking. The concept of spontaneous symmetry breaking is a form of symmetry breaking, where the system itself is still invariant under a symmetry transformation, however, a selected ground state seems to “break” this invariance. This gives rise to interesting and important phenomena. Thus, the relevant fundamental definitions and processes for spontaneously broken symmetry transformations of physical systems are discussed in the respective subsections below. The focus of this discussion lies on the investigation of spontaneous symmetry breaking of continuous global and local gauge transformations, two of the most important and common symmetry transformations occurring in nature.

The description of spontaneously broken global symmetries is provided by the Goldstone theorem and leads to the occurrence of massless fields, corresponding to so-called Goldstone bosons. Whilst, for local gauge symmetries, this description of spontaneous symmetry breakdown is called the Higgs mechanism, which gives rise to a field, called the Higgs field together with a particle, called the Higgs bosons. Furthermore, arguably the most important result of the Higgs mechanism is that it allows some gauge bosons to acquire mass.
II. GAUGE PRINCIPLE

In this section the gauge principle will be investigated, since its results and concepts are vital to the rest of the discussion of the topics mentioned in the introduction.

The gauge principle is essentially the extension of a global gauge invariance to a local gauge invariance. As an example one can look at the U(1) symmetry from which the electromagnetic coupling arises [2].

One starts from the rather simple Lagrangian for a complex scalar field $\phi(x)$:

$$L_0(\phi(x), \partial_\mu \phi(x)) = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi).$$  \hspace{1cm} (1)

The Lagrangian density in equation (1) is invariant under a constant phase change of $\phi(x)$:

$$\phi(x) \rightarrow e^{-i\alpha} \phi(x), \quad \alpha \in \mathbb{R}.$$  \hspace{1cm} (2)

Here (2) indicates a global gauge transformation of the symmetry group U(1). For the case of a local U(1) gauge symmetry the parameter $\alpha$ becomes a function and therefore depends on the position in space-time:

$$\alpha \rightarrow \alpha(x) \Rightarrow \phi(x) \rightarrow e^{-i\alpha(x)} \phi(x).$$  \hspace{1cm} (3)

The gauge transformations at different points in space-time are independent of each other. However, using this local gauge transformation the Lagrangian density in equation (1) is no longer invariant. $\partial_\mu \phi(x)$ acquires an extra term:

$$\partial_\mu \phi(x) \rightarrow e^{-i\alpha(x)} \partial_\mu \phi(x) + \phi(x) \partial_\mu e^{-i\alpha(x)}.$$  \hspace{1cm} (4)

To overcome this extra term and to ensure a locally gauge invariant transformation a new field $A_\mu$ needs to be defined, the gauge field:

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x).$$  \hspace{1cm} (5)

Furthermore, the derivative on the field will be replaced by the so-called covariant derivative:

$$D^\mu \phi(x) = [\partial^\mu + ieA^\mu(x)] \phi(x).$$  \hspace{1cm} (6)

If one plugs (5) and (6) into equation (1) it is easy to see that $L_0(\phi(x), D^\mu \phi(x))$ is invariant under the local gauge transformation (3). One can construct a gauge invariant Lorentz scalar $F^{\mu\nu} F_{\mu\nu}$ from $A_\mu$, where:

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x).$$  \hspace{1cm} (7)
is the field strength tensor of the gauge field. With this Lorentz scalar, which can be regarded as the kinetic term for the gauge field, the Lagrangian describes a closed dynamical system. This results in the full description of the local U(1) gauge invariant Lagrangian density:

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi^\dagger \phi),
\]

which can be thought of as a model for electrodynamics.

III. SPONTANEOUS SYMMETRY BREAKING

There are two methods for symmetry breaking [5]. One process is called “explicit” breaking of a symmetry. In this process an outside factor, for example a force, actively breaks the symmetry in the form of adding extra terms to the equation of motions or the Lagrangian that make the whole system invariant under previous symmetries.

The other process of symmetry breaking is called “spontaneous symmetry breaking”. If the potential of a system, invariant under a symmetry transformation, admits a constant non-trivial lowest energy classical configuration for the Lagrangian, the process of selecting a ground state from the lowest-energy classical configuration, essentially a set of degenerate ground states, is called “Spontaneous symmetry breakdown”. Every “choice” for the ground state is equal and, to be precise, does not break the symmetry. The symmetry transformation that seems to be lost, is “hidden” in the relations of the degenerate ground states. So under a symmetry transformation the ground state gets transformed to another ground state, not necessarily to the same state but to a state that also minimizes the potential (system). So the system in itself is still invariant under the symmetry, however the ground state is not.

A. Linear sigma model

As an illustrative example for spontaneous symmetry breaking of a global continuous symmetry, we investigate the linear sigma model [1]. For a system with \( N \) real scalar fields \( \phi^i \), where \( i = 1, \ldots, N \), the Lagrangian has the following form:

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^i)^2 + \mu^2 (\phi^i)^2 - \frac{\lambda}{2} [(\phi^i)^2]^2.
\]
FIG. 1: Potentials illustrating the $\mu^2 < 0$ case in (a) and $\mu^2 > 0$ in (b).

This Lagrangian is invariant under the global continuous transformation:

$$\phi^i \rightarrow R^{ij} \phi^j,$$

where $R$ is a $N \times N$ orthogonal matrix. Therefore, the symmetry group of the system is $O(N)$. The potential was chosen to have the following form:

$$V = -\mu^2 (\phi^i)^2 + \frac{\lambda}{2}[(\phi^i)^2]^2,$$

where $\lambda, \mu^2$ are real parameters. This results in different descriptions for the system depending on the values for $\lambda$ and $\mu^2$. There exist theories only for $\lambda > 0$, because for $\lambda < 0$ there would be no lower bound for the energy of the system, thus there are two remaining cases, depending on the sign of $\mu^2$. For a potential with $\mu^2 < 0$, corresponding to Fig.(1.a), no spontaneous symmetry breaking can occur, because the ground state is definite and is located in the origin. However, the situation for $\mu^2 > 0$, illustrated in Fig.(1.b), is very different. Here, the ground state is degenerate and this results in a spontaneous symmetry breaking, which can additionally be seen in the following derivation. The lowest-energy classical configuration

$$\langle \phi^i_0 \rangle^2 = \frac{\mu^2}{\lambda},$$

minimizes the potential. However, the above relation only determines the length for a ground state $\phi^i_0$. The direction in which the ground state points can be chosen arbitrarily, as long as (12) is satisfied. For convenience the following ground state is selected:

$$\phi^i_0 = (0, \ldots, 0, v) \quad \text{for} \quad v = \sqrt{\frac{\mu^2}{\lambda}}.$$
It should be emphasised that one could also have selected a different ground state and the following derivation would be similar. This is exactly the process of spontaneous symmetry breaking, since all choices for the ground state are equal.

To investigate the behaviour of the system near the selected ground state, one can find a linear expansion near the ground state:

$$\phi^i = \begin{pmatrix} \pi^k(x) \\ v + \sigma(x) \end{pmatrix} \text{ for } k = 1, \ldots, N - 1 .$$  \hspace{1cm} (14)

By inserting this into the Lagrangian (9), one obtains:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2(x) + \mathcal{O}((\sigma, \pi)^3) .$$  \hspace{1cm} (15)

The above Lagrangian consists of \((N - 1)\) massless \(\pi(x)\) fields, where the mass terms cancel in the potential, and one massive \(\sigma(x)\) field with mass \(m_\sigma = \sqrt{2}\mu\). Terms of higher order, as well as mixed terms, were neglected, since the fields are presumed to be small. The original \(O(N)\) symmetry is now not clearly visible any more. It is “hidden” in a \(O(N - 1)\) subgroup and the relations of the degenerate ground states, since any choice is possible. The \(O(N - 1)\) subgroup rotates the remaining \((N - 1)\) massless \(\pi(x)\) fields among themselves. The number of massless fields corresponds to the number of generators of the spontaneously broken continuous global symmetry and are called Goldstone bosons:

$$O(N) - O(N - 1) = N - 1 .$$  \hspace{1cm} (16)

This is a result of the Goldstone theorem.

**IV. GOLDSTONE BOSON AND THEOREM**

**A. Goldstone theorem**

The Goldstone theorem was formulated for system invariant under a global continuous symmetry, which is spontaneously broken \([1]\). The selected ground state is not invariant under the action of the symmetry any more. The theorem states that for every generator of a spontaneously broken symmetry, there will appear a massless (and spinless) field term in the Lagrangian, called the Goldstone boson. The symmetry is said to be in the “Goldstone
mode”.

A proof of the Goldstone theorem for classical field theories:

For a set of scalar fields $\phi^a(x)$ the Lagrangian has the following form:

$$\mathcal{L} = \text{(derivatives)} - V(\phi) .$$

(17)

The minimum for the above potential is a constant field $\phi_0^a$, for which:

$$\frac{\partial}{\partial \phi^a} \bigg|_{\phi^a=\phi_0^a} V(\phi) = 0 .$$

(18)

Now the Taylor expansion of the potential around this minimum is:

$$V(\phi) = V(\phi_0^a) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b \left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right)_{\phi_0} + ...$$

(19)

In the Taylor expansion

$$\left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right)_{\phi_0} = m_{ab}^2$$

(20)

is a symmetric matrix with mass terms for the fields as eigenvalues, which are all non-negative since it is evaluated at the minimum $\phi_0$.

Now to prove the Goldstone theorem, one needs to show that every spontaneously broken symmetry transformation results in a zero eigenvalue of the mass matrix.

A general continuous symmetry transformation is of the form:

$$\phi^a \rightarrow \phi^a + \alpha \Delta^a(\phi),$$

(21)

for $\alpha$ an infinitesimal parameter and $\Delta^a$ a function of all $\phi^a$.

Since for constant fields the derivative terms in the Lagrangian vanish, only the potential needs to be invariant under the above transformation:

$$V(\phi^a) = V(\phi^a + \alpha \Delta^a(\phi)) \Leftrightarrow \Delta^a(\phi) \frac{\partial V(\phi)}{\partial \phi^a} = 0 .$$

(22)

After differentiating the second relation in equation (22) with respect to $\phi^b$ and evaluating $\phi$ at $\phi_0$, one obtains:

$$0 = \left( \frac{\partial \Delta^a(\phi)}{\partial \phi^b} \right)_{\phi_0} \left( \frac{\partial V(\phi)}{\partial \phi^a} \right)_{\phi_0} + \Delta^a(\phi_0) \left( \frac{\partial^2 V(\phi)}{\partial \phi^a \partial \phi^b} \right)_{\phi_0} .$$

(23)
From this, it follows that
\[ 0 = \Delta^a(\phi_0) \left( \frac{\partial^2 V(\phi)}{\partial \phi_a \partial \phi^b} \right)_{\phi_0} . \]  \hspace{1cm} (24)

If the continuous symmetry leaves the selected ground state \( \phi_0 \) invariant \( \Rightarrow \Delta^a(\phi_0) = 0 \).
However, for the case of spontaneously symmetry breaking:
\[ \Delta^a(\phi_0) \neq 0 \Rightarrow \left( \frac{\partial^2 V(\phi)}{\partial \phi_a \partial \phi^b} \right)_{\phi_0} = 0 , \]  \hspace{1cm} (25)
which means that \( \Delta^a(\phi_0) \) is the desired eigenvector corresponding to a zero eigenvalue. \( \square \)

V. HIGGS MECHANISM

A. General case

For local gauge symmetries that are spontaneously broken, no Goldstone bosons appear. However, the massless gauge bosons acquire mass and a longitudinal polarisation. A system undergoing such a phenomenon is said to be in the “Higgs mode”.

The Higgs mechanism can be described in a general way [1]. Suppose there is a set of scalar fields \( \phi_i \) that appear in a Lagrangian invariant under a group of continuous symmetries \( G \):
\[ \phi_i \rightarrow (1 + i\alpha^a t^a)_{ij} \phi_j . \]  \hspace{1cm} (26)
The scalar fields will be written as real-valued fields, so for \( N \) complex fields there will be \( 2N \) real fields that describe the system. From this it follows that the \( t^a \) must be purely imaginary and antisymmetric, since they need to be hermitian, i.e.:
\[ t^a_{ij} = iT^a_{ij} , \]  \hspace{1cm} (27)
for \( T^a \) real and antisymmetric. Now one promotes \( G \) to a local gauge symmetry, for example by applying the gauge principle:
\[ D_\mu \phi = (\partial_\mu - igA^a_\mu t^a)\phi = (\partial_\mu + gA^a_\mu T^a)\phi . \]  \hspace{1cm} (28)
The kinetic term in the Lagrangian becomes:
\[ \frac{1}{2}(D_\mu \phi)^2 = \frac{1}{2}(\partial_\mu \phi)^2 + gA^a_\mu (\partial_\mu \phi_i T^a_{ij} \phi_j) + \frac{1}{2}g^2 A^a_\mu A^b_\mu (T^a \phi)_i (T^b \phi)_i . \]  \hspace{1cm} (29)
Now let $\phi_i$ acquire a non-trivial lowest-energy configuration and expand the fields around a specific ground state, satisfying this condition. This leads to the following structure of the gauge boson mass:

$$\Delta \mathcal{L} = \frac{1}{2} m_{ab}^{2} A_{\mu}^{a} A^{b\mu}.$$  

$$\Rightarrow m_{ab}^{2} = g^{2} (T^{a} \phi_{0})_{i} (T^{b} \phi_{0})_{i}.$$  

The mass matrix $m_{ab}^{2}$ is positive semi-definite since any diagonal element satisfies:

$$m_{aa}^{2} = g^{2} (T^{a} \phi_{0})_{i}^{2} \geq 0.$$  \hspace{1cm} (30)

If $T^{a}$ leaves the ground state invariant: $T^{a} \phi_{0} = 0$ and the generator does not contribute to the mass matrix, which means that the corresponding gauge boson will be massless. For the generators of a spontaneously broken symmetry, the gauge boson acquires a mass.

### B. Abelian example

An Abelian example for the Higgs mechanism is the Lagrangian of a complex scalar field in equation (8) with the potential [1, 2]:

$$V(\phi) = -\mu^{2} (\phi^{\dagger} \phi) + \frac{\lambda}{2} (\phi^{\dagger} \phi)^{2}, \quad \mu^{2} > 0, \lambda > 0 \quad \text{for} \quad |\phi_{0}|^{2} = \frac{\mu^{2}}{\lambda}.$$  \hspace{1cm} (31)

This Lagrangian is invariant under the local $U(1)$ transformation described in equations (3), (5) and (6). Using the above local gauge invariance of the Lagrangian and the fact that one can write a complex scalar field in terms of two real scalar fields:

$$\phi(x) = \eta(x) e^{i\kappa(x)}$$  \hspace{1cm} (32)

it is always possible to find a local gauge transformation, called a unitary gauge, in which the field $\phi(x)$ becomes real-valued at every point in space-time. Therefore, if one selects a definite ground state $\phi_{0}$ from the lowest-energy configuration of the system $|\phi_{0}|^{2}$ and expands the field $\phi$ around it, one obtains:

$$\phi(x) = \phi_{0} + \eta(x).$$  \hspace{1cm} (33)
The potential transforms to the following:

\[ V(\phi) = -\frac{\mu^2}{2}|\phi_0|^2 + \frac{1}{2}(2\mu^2)\eta(x)^2 + \mathcal{O}(\eta^3) \]  

(34)

Here the first term is simply a constant and terms of higher order are presumed to be very small.

The kinetic term however gives rise to an additional term:

\[ |D_\mu \phi|^2 = (D^\mu \phi(x))^\dagger (D_\mu \phi(x)) \]

\[ = \frac{1}{2}(D^\mu \eta(x))^\dagger (D_\mu \eta(x)) + e^2 \phi_0^2 A_\mu A_\mu + \ldots \]

\[ =: \Delta \mathcal{L} \]  

(35)

The \( \Delta \mathcal{L} \) term is called a photon mass term with mass \( m^2_A = 2e^2 \phi_0^2 \), which arises from the non-trivial lowest-energy configuration of the field.

The full Lagrangian, up to order two in the expanded fields, can be seen in the following:

\[ \mathcal{L} = -\frac{1}{4}F_{\mu \nu}F^{\mu \nu} + e^2|\phi_0|^2 A_\mu A^\mu + (D_\mu \eta)(D^\mu \eta) + \frac{1}{2}(2|\phi_0|^2\lambda)\eta^2 + \mathcal{O}(\eta^3) \]  

(36)

From the above example it follows that gauge bosons can only acquire mass when they are associated with such a non-trivial lowest-energy configuration. In four dimensions this arises from a massless scalar field, which are available in spontaneously broken symmetries. In quantum field theory these bosons carry with them the right quantum numbers from the broken symmetry current to appear as intermediate states in the vacuum. This behaviour is explained by the gauge field \( A_\mu \) coupling directly to the Goldstone bosons.

Landau and Ginzberg, whilst trying to describe the theory of superconductors coupling to an external electromagnetic field, found the same Lagrangian for their system as in equation (8). The coupling of two electrons, as so called Cooper-pairs, is the reason for the appearance of spontaneously symmetry breaking in the description of a superconductor. As has been shown in the derivation above, the initially massless gauge boson, acquires mass due to the photon mass term \( \Delta \mathcal{L} \). This leads to the gauge boson, photon, of the external electromagnetic field to have only a finite penetration depth, of \( r = \frac{1}{m_A} \), into the field of the superconductor. This explains the Meissner effect, which states the exclusion of macroscopic magnetic fields from a superconductor.

It has to be stated here that there is no fundamental Higgs boson in this effect. This phenomenon is called “dynamical symmetry breaking” and usually coupled fermions, here
electrons, behave like bosons and these take over the role of the Higgs boson in providing mass for the initially massless gauge bosons.

C. Non-Abelian example

In this section, the non-Abelian case will be illustrated for the bosonic sector of the standard electroweak theory of Glashow-Weinberg-Salam [3]. The gauge group is \( SU(2) \times U(1) \) with gauge coupling constants \( g \) and \( g' \), respectively. \( A^a_\mu \) (\( a = 1,2,3 \)) is the gauge field of \( SU(2) \) and \( B_\mu \) for \( U(1) \). The field \( \phi \) is a doublet of scalar fields and has a \( U(1) \) charge, also called weak hypercharge, of \( Y = \frac{1}{2} \). The Lagrangian of this theory is:

\[
\mathcal{L} = -\frac{1}{4} F^a_\mu F^{a\mu} - \frac{1}{4} B_\mu B^\mu + (D_\mu \phi)^\dagger (D^\mu \phi) - \lambda \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2 \tag{37}
\]

for

\[
F^a_\mu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_\epsilon^{abc} A^b_\mu A^c_\nu \tag{38}
\]

\[
B_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu
\]

and

\[
D_\mu \phi = \partial_\mu \phi - ig_2 \tau^a A^a_\mu \phi - ig'_2 B_\mu \phi . \tag{39}
\]

For the field \( \phi \) the chosen ground states will be \( \phi^{(v)} \):

\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \phi^{(v)} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \tag{40}
\]

\( A^a_\mu = B_\mu = 0 \)

for \( v \) a constant. As for the generators of the electroweak theory, they consist of the hermitian matrices \( T^a = \frac{\tau^a}{2} \), where the \( \tau^a \) are the Pauli matrices, and \( Y = \frac{1}{2} \cdot 1_{2\times2} \). To find the unbroken generators \( Q \) the gauge fields will be ignored. \( Q \) should also be hermitian and satisfy:

\[
Q \phi^{(v)} = 0 . \tag{41}
\]

Therefore \( Q \) is of the form:

\[
Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv T^3 + Y . \tag{42}
\]
This $Q$ is the generator of an unbroken $U(1)_{\text{e.m.}}$ symmetry, left over after the symmetry $SU(2) \times U(1)$ breaks for the ground states defined above. It corresponds to a massless gauge field, which will be identified with the electromagnetic field. The electromagnetic potential $A_\mu$ is a linear combination of the fields $A_\mu^a$ and $B_\mu$. Now for small perturbations of the fields about the vacuum state, in unitary gauge and $\chi(x)$ a real scalar field, one finds:

$$\tilde{\phi} = \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} + \frac{\chi(x)}{\sqrt{2}} \end{array} \right).$$  \tag{43}$$

The covariant derivative results in the following expression:

$$D_\mu \tilde{\phi} = \left( \begin{array}{c} -\frac{ig}{2\sqrt{2}} (A_\mu^1 - iA_\mu^2) (v + \chi(x)) \\ \frac{1}{\sqrt{2}} \partial_\mu \chi(x) - \frac{i}{2\sqrt{2}} (g' B_\mu - g A_\mu^3) (v + \chi(x)) \end{array} \right).$$  \tag{44}$$

We introduce complex:

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (A_\mu^1 \mp iA_\mu^2)$$  \tag{45}$$

and two real fields:

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (gA_\mu^3 - g' B_\mu)$$  \tag{46}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (gB_\mu + g' A_\mu^3).$$

From these it follows that the covariant derivative only contains the field $Z_\mu$ and the following property is satisfied:

$$Z_\mu^2 + (A_\mu)^2 = (A_\mu^3)^2 + B_\mu^2.$$  \tag{47}$$

To second order in the fields defined above the covariant derivative term in the Lagrangian becomes:

$$\left[ (D_\mu \tilde{\phi})^* D^\mu \tilde{\phi} \right]^{(2)} = \frac{1}{2} (\partial_\mu \chi(x))^2 + \frac{g^2 v^2}{4} W_\mu^+ W^-\mu + \frac{1}{2} \left( \frac{(g^2 + g'^2) v^2}{4} \right) Z_\mu^2.$$  \tag{48}$$

Similarly the kinetic terms of the vector fields up to quadratic order results in:

$$-\frac{1}{4} F_{\mu\nu} a F^{a\mu\nu} - \frac{1}{4} B_\mu^2 = -\frac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} - \frac{1}{4} (F_{\mu\nu}^3)^2 - \frac{1}{4} (B_{\mu\nu})^2,$$  \tag{49}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$$

$$W_{\mu\nu}^\pm = \partial_\mu W_{\nu}^\pm - \partial_\nu W_{\mu}^\pm.$$  \tag{50}$$
Using the relation (47), one can rewrite the right hand side of equation (49) as:

\[
-\frac{1}{2} \mathcal{W}_{\mu\nu}^+ \mathcal{W}^{-\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

where

\[
Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

The quadratic part of the potential becomes:

\[
\lambda v^2 \chi(x)^2.
\]

Together with the following notation:

\[
m_W = \frac{g v}{2}
\]

\[
m_Z = \frac{\sqrt{g^2 + g'^2} v}{2}
\]

\[
m_\chi = \sqrt{2} \lambda v
\]

the full new Lagrangian of the quadratic terms is of the form:

\[
\mathcal{L}^{(2)} = -\frac{1}{2} \mathcal{W}_{\mu\nu}^+ \mathcal{W}^{-\mu\nu} + m_W^2 W_\mu^+ W^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
- \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{m_Z^2}{2} Z_\mu Z^\mu
+ \frac{1}{\sqrt{2}} (\partial_\mu \chi(x))^2 + \frac{m_\chi^2}{2} \chi(x)^2.
\]

This Lagrangian contains a massive complex vector field \( W_{\mu\nu}^\pm \) with mass \( m_W \) (W-boson field), a massless vector field (photon field \( A_\mu \)), a massive real vector field \( Z_\mu \) with mass \( m_Z \) (Z-boson field) and a massive real scalar field \( \chi(x) \) (Higgs boson field).

In particle physics there also exists the concept of the weak mixing angle \( \theta_W \), defined in terms of the coupling constants in the following way:

\[
\cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}}
\]

\[
\sin \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}.
\]

Using these relations the defined fields in equation (46) become:

\[
Z_\mu = \cos \theta_W A_\mu^3 - \sin \theta_W B_\mu
\]

\[
A_\mu = \cos \theta_W B_\mu + \sin \theta_W A_\mu^3.
\]
Also the masses of the $W$- and $Z$-bosons are related to one another by

$$m_Z = \frac{m_W}{\cos \theta_W}.$$  \hfill (58)

Measurements of these masses and $\theta_W$ independently show the good accuracy of this relation in nature, with experimentally determined values of $\sin^2 \theta_W = 0.23$, $m_Z = 91$ GeV, $m_W = 80$ GeV.

VI. SPONTANEOUS SYMMETRY BREAKING IN THE CONTEXT OF GROUP THEORY

A. Spontaneous symmetry breaking

For Lagrangians with scalar potentials spontaneous symmetry breaking is characterized in the following way [2, 4]:

The potential $V(\phi)$ of a scalar field $\phi(x)$ is invariant under a continuous unitary representation $U(g)$ of the internal symmetry group $G$:

$$V(U(g)\phi) = V(\phi).$$ \hfill (59)

If the minimum of the potential $\phi_0$ is not invariant under the group

$$U(g)\phi_0 \neq \phi_0$$ \hfill (60)

for some $g \in G$, the symmetry is said to be spontaneously broken. As an example the field $\phi$ with the potential:

$$V(\phi) = \lambda \left( \phi^2 - c^2 \right)^2$$ \hfill (61)

belongs to a $n$-dimensional representation of $\text{SO}(n)$. Here $c$ and $\lambda$ are constants. The minimum of the potential occurs at a non-symmetrical point $\phi_0 = c\hat{n}$, for $\hat{n}$ a unit vector.

B. The little group

The symmetry may not be broken completely, so there may exist some $g \in G$ for which the ground state is still invariant. These $g$ form a subgroup, called $H \subset G$, also referred to as the little group of the ground state $\phi_0$:

$$U(h)\phi_0 = \phi_0 \Leftrightarrow h \in H.$$ \hfill (62)
For the above example, $G = \text{SO}(n)$ and $H = \text{SO}(n-1)$. It can be seen that this ground state is not unique. The whole orbit $U(g)\phi_0$ will minimize the potential. Therefore the potential minimum is degenerate, which relates to an orbit consisting of more than one point. This orbit has physical meaning for gauge symmetries. The little group is the same for all points on the orbit, since:

$$U(h)\phi_0 = \phi_0 \Rightarrow U(ghg^{-1})U(g)\phi_0 = U(g)\phi_0 .$$

(63)

This leads to a definition of spontaneous symmetry breaking specified by the orbit of the potential minimum. The Lie algebra of the little group forms a subset $\{l_{\alpha}\}$, for ($\alpha = 1, ..., N-K$), of the Lie algebra of $G$, its generators denoted by $\{L_a\}$ with ($a = 1, ..., N$), with the Lie bracket:

$$[l_{\alpha}, l_{\beta}] = iC_{\alpha\beta\gamma}l_{\gamma} .$$

(64)

The elements of the Lie algebra of $G$ are split into two disjoint subsets $\{L_j, l_{\alpha}\}$. Elements that leave the minimum invariant $l_{\alpha}\phi_0 = 0$ and $L_j\phi_0 \neq 0$. These generators will be represented by real antisymmetric matrices:

$$T_j = -iL_j , \quad j = 1, ..., K$$

$$l_{\alpha} = -il_{\alpha} , \quad \alpha = 1, ..., N - K .$$

(65)

The minimum $\phi_0$ has $R$ real components. A scalar product in this space is denoted by:

$$(f, Og) = \sum_{n=1}^{R} \sum_{m=1}^{R} f_nO_{nm}g_m .$$

(66)

From this it follows that $(T_j\phi_0, T_j\phi_0)$ is a real symmetric matrix with positive-definite eigenvalues. Therefore, the $T_j\phi_0$ form a $K$-dimensional subspace of the $R$-dimensional representational vector space. This space is called the Goldstone space and its complement of dimension $R - K$ is called the Higgs space. For any vector $\phi$ in the $R$-dimensional representational vector space, and for a compact gauge group $G$, there always exists a gauge transformation $U_0$ such that

$$(T_j\phi_0, U_0\phi) = 0 , \quad j = 1, ..., K .$$

(67)

This is referred to as unitary gauge. Therefore, since the above relation holds for every point in space-time, if $\phi(x)$ is a solution, it must be a continuous function of $x$. 
C. The Higgs mechanism

From the above it follows that fields near the vacuum solution in unitary gauge are of the form:

\[
\phi(x) = \begin{pmatrix} 0 \\ \tilde{\phi}_0 + \eta(x) \end{pmatrix}
\]

Goldstone space, $K$-dim.

\[
A_\alpha^\mu(x) = \text{small}.
\]

(68)

To first order one finds the following quantities:

\[
V(\phi) = \frac{1}{2}(\eta, V''(\phi_0)\eta)
\]

(69)

\[
j_\alpha^\mu = -g^2(T_a\phi_0, T_b\phi_0)A_\alpha^\mu
\]

(70)

with which one can define the following mass matrices:

\[
(\mu^2)_{rs} = \begin{pmatrix} 0 & 0 \\ 0 & V''(\phi_0) \end{pmatrix}
\]

(71)

\[
(M^2)_{ab} = g^2(T_a\phi_0, T_b\phi_0) = \begin{pmatrix} (M^2)_{ij} & 0 \\ 0 & 0 \end{pmatrix}.
\]

(72)

The linearised equations for the system then have the following form

\[
\Box^2 \eta_r + (\mu^2)_{rs} \eta_s = 0 \quad (r = 1, \ldots, R - K)
\]

\[
\Box^2 A_i^\nu + (M^2)_{ij} A_j^\nu = 0 \quad (i = 1, \ldots, K)
\]

\[
\Box^2 A_\alpha^{\mu \nu} - \partial^\nu (\partial_\mu A_\alpha^{\nu}) = 0 \quad (\alpha = 1, \ldots, N - K).
\]

(73)

The results are summarised in the following table:

<table>
<thead>
<tr>
<th>Field</th>
<th>no. of Fields</th>
<th>no. of independent components</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_r$ (Higgs, massive)</td>
<td>$R - K$</td>
<td>$R - K$</td>
</tr>
<tr>
<td>$A_i^\mu$ (gauge, massive)</td>
<td>$K$</td>
<td>$3K$</td>
</tr>
<tr>
<td>$A_\alpha^\mu$ (gauge, massless)</td>
<td>$N - K$</td>
<td>$2(N - K)$</td>
</tr>
</tbody>
</table>

$G$ is spontaneously broken down to $H$

$N$ = number of generators of $G$

$N - K$ = number of generators of $H$

$R$ = dimension of real representation of $G$
VII. CONCLUSION

To conclude, the investigation of spontaneously broken symmetries and the concepts used to describe them provided insights into how the electroweak interaction is broken down to the weak and the electromagnetic interactions, as well as give an explanation of the Meissner effect for superconductors. The Goldstone theorem and the Higgs mechanism were discussed in terms of classical field theory, however their applications are not limited to it and the general approaches, which were mentioned, but were beyond the scope of detailed discussion, can be of use in many other aspects of field theories.
VIII. APPENDIX

A. Mathematical definitions

In the following definitions, $G$ is a group, $M$ a set and $\alpha : G \times M \to M$ is a group action of $G$ on $M$.

The orbit $M_m$ of a point $m \in M$ is the set

$$M_m := \alpha(G, m) = \{\alpha(a, m); a \in G\}.$$  \hfill (74)

The stabiliser $G_m$ (also called little group) of a point $m \in M$ is the subgroup of $G$ which leaves the point $m$ fixed

$$G_m := \{a \in G; \alpha(a, m) = m\}.$$  \hfill (75)

These definitions have been taken from [6].

B. Summary Sheet

GENERAL FORMULAS

Global U(1) invariant Lagrangian density:

$$\mathcal{L}_0(\phi(x), \partial_\mu \phi(x)) = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi)$$ \hfill (76)

Global transformation:

$$\phi(x) \to e^{-i\alpha(x)} \phi(x) \quad \alpha \in \mathbb{R}$$ \hfill (77)

Local gauge transformation:

$$\phi(x) \to e^{-i\alpha(x)} \phi(x) \quad \text{for } \alpha(x) \text{ a real function}$$ \hfill (78)

Gauge field under local gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x)$$ \hfill (79)

Covariant derivative:

$$D^\mu \phi(x) = [\partial^\mu + ieA^\mu(x)] \phi(x)$$ \hfill (80)

Field strength tensor:

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$$ \hfill (81)
Local U(1) gauge invariant Lagrangian density:

\[ \mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} + (D^\mu \phi(x))^\dagger (D_\mu \phi(x)) - V(\phi^\dagger \phi) \]  

**GAUGE PRINCIPLE**

Extension of a global to a local gauge transformation.

**SPONTANEOUS SYMMETRY BREAKING**

Process of selecting a ground state from a non-trivial lowest-energy configuration of a system, which is no longer invariant under full action of the symmetry group. Symmetry is not lost, merely "hidden" in the relations of the degenerate ground states.

Potential:

\[ V(\phi^\dagger \phi) = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{2} (\phi^\dagger \phi)^2 \text{ for } \mu^2 > 0, \lambda > 0 \text{ and } |\phi_0|^2 = \frac{\mu^2}{\lambda} \]  

**Goldstone theorem**

For every generator of a continuous global symmetry that is spontaneously broken, there appears a massless field in the Lagrangian, corresponding to a Goldstone boson.

**Higgs mechanism**

A spontaneously broken local gauge symmetry can generate mass for the gauge bosons.

**Unitary gauge**

In a U(1) gauge theory, a unitary gauge, is a local U(1) gauge transformation, such that the field is real-valued at every point in space-time.
IX. REFERENCES


Additional Reading: