CURVATURE OF PIECEWISE FLAT SPACES

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Introduction.

In [Regge, 1961], T. Regge associated to a piecewise flat manifold, X^n, a measure, $R(X^n)$, supported on the $(n-2)$-skeleton.

He proposed that it should play the role of $R(M^n)$, scalar curvature times the volume measure for smooth riemannian manifolds M^n.

Physicists who then discussed "Regge calculus", seemed to take for granted: If a sequence X_i^n converges to M^n in a suitable sense, then in a suitable sense, $R(X_i^n) \to R(M^n)$.

For $n > 2$, this was first precisely formulated and rigorously proved by Cheeger-Muller-Schrader; see [CMS, 1984].

Piecewise flat spaces.

Though had in mind applications to general relativity, here we only consider the Euclidean case.

Definition. A *piecewise flat space*, X^n, is a triangulated n-manifold, perhaps with nonempty boundary, and a length metric space whose restriction to each n-simplex σ^n is isometric to the convex hull of a collection of $n+1$ points in general position in $\mathbb{R}^n$.

If we fix the combinatorial structure, we can regard the metric as a function of the squares of the edge lengths.

However, our curvature invariants are independent of the triangulation and depend only on the metric structure.

Regge's scalar curvature measure $R(X^n)$.

Consider an *interior* $(n-2)$-simplex $\sigma^{n-2} \subset \mathrm{int}(X^n)$.

Denote by $|\sigma^{n-2}|$ the area measure of σ^{n-2} and define the normal "angle defect" by

$$P_\chi(C^\perp(\sigma^{n-2})) := 1 - \frac{\theta(\sigma^{n-2})}{2\pi}\,.$$

Here, $\theta(\sigma^{n-2})$ denotes the sum over all $\sigma^n \supset \sigma^{n-2}$, of the corresponding dihedral angles and $C^\perp(\sigma^{n-2})$ denotes the normal cone.

Define the Regge scalar curvature measure by:

$$R(X^n) := \sum_{\sigma^{n-2} \subset \mathrm{int}(X^n)} P_\chi(C^\perp(\sigma^{n-2})) \cdot |\sigma^{n-2}|\,.$$

Boundary curvature $H(X^n)$.

There is also a mean curvature measure supported on ∂X^n:

$$H(X^n) := \sum_{\sigma^{n-2} \subset \partial X^n} P_\chi(C^\perp(\sigma^{n-2})) \cdot |\sigma^{n-2}| \, ,$$

where

$$P_\chi(C^\perp(\sigma^{n-2})) := 1 - \frac{\theta(\sigma^{n-2})}{\pi} \, .$$

$H(X^n)$ is the analog of $H(M^n)$, the mean curvature of the ∂M^n times the riemannian volume of ∂M^n.

Formal properties.

Note: $R(X^n)$, $H(X^n)$, are independent of the triangulation.

They depend only on the *metric* and share the behavior of $R(M^n), H(M^n)$, under scaling and products.

For $n = 2$, a version of the Gauss-Bonnet theorem holds.

Lipschitz-Killing curvature measures.

Actually, a more general convergence theorem for so-called *Lipschitz-Killing curvature measures* R^i and their associated boundary measures H^i was given in [CMS, 1984].

Here, i is a nonnegative integer; $R^i \equiv 0$ for i odd or $i > n$.

In terms of an orthonormal frame field, $R^i(M^n)$ is a certain invariant polynomial of degree $i/2$ in the components of the curvature tensor, multiplied by the riemannian measure.

The expressions for $H^i(M^n)$ also involve components of the second fundamental form; they need not vanish if i is odd.

Particular cases.

- $R^0(M^n)$, the riemannian volume.

- $R^2(M^n) := R(M^n)$, scalar curvature times riemannian volume.

- $R^{2m}(M^{2m})$, the Chern-Gauss-Bonnet integrand times riemannian volume.

- H^0, the riemannian volume of ∂M^n.

- $H(M^n) := H^2(M^n)$, the mean curvature times the riemannian volume of ∂M^n.

- $H^n(M^n)$, the boundary Chern-Gauss-Bonnet integrand, times riemannian volume of ∂M^n.

Role of Lipschitz Killing curvatures.

Apart from Chern-Gauss-Bonnet, Lipschitz-Killing curvatures arise in other contexts including:

- Weyl's formula for the volume of tubes.
- Kinematic formulas.
- Asymptotic expansions for traces of heat kernels.

Axiomatic characterization.

The proof of the convergence theorem in [CMS, 1984] made essential use of Gilkey's characterization of $R^i(M^n)$ among polynomials of degree $i/2$ in curvature:

- Invariance under change of orthonormal basis.
- Vanishing for isometric products $M^{2i+1} \times \mathbb{R}^{n-2i+1}$,
- Value for $S^2 \times \cdots \times S^2 \times \mathbb{R}^{n-2i}$.

Gilkey's corresponding characterization for $H^i(M^n)$ was also used for the convergence theorem for the boundary curvatures; see [Gilkey, 1974, 1975].

Lipschitz-Killing curvatures; piecewise flat case.

In the piecewise flat case, define:

$$R^i(X^n) := \sum_{\sigma^{n-i} \subset \mathrm{int}(X^n)} P_\chi(C^\perp(\sigma^{n-i})) \cdot |\sigma^{n-i}| \,.$$

$$H^i(X^n) := \sum_{\sigma^{n-i} \subset \partial X^n} P_\chi(C^\perp(\sigma^{n-i})) \cdot |\sigma^{n-i}| \,.$$

The generalized defect angles $P_\chi(C^\perp(\sigma^{n-i}))$ involve certain *products* of dihedral angles.

$P_\chi(C^\perp(\sigma^{n-i}))$ is the Chern-Gauss-Bonnet measure for the normal cone to σ^{n-i}, where $\dim C^\perp(\sigma^{n-i}) = i$.

Role in the piecewise flat case.

The piecewise flat Lipschitz-Killing curvatures, $R^i(X^n)$, $H^i(X^n)$, have the same scaling behavior and behavior under products as their counterparts in the smooth case.

As in the smooth case, they had previously arisen in connection with volumes of tubes, kinematic formulas and heat kernel asymptotics.

As indicated, there is a piecewise flat Chern-Gauss-Bonnet formula

$$\chi(X^n) = R^n(X^n) + H^n(X^n)\,.$$

$R^n(X^n)$, $H^n(X^n)$, are supported on the 0-skelton of X^n.

Regge's Lemma.

Given a 1-parameter family of flat metrics σ_t^n on the n-simplex, let $\theta(\sigma_t^{n-2})$ denote the associated family of dihedral angles for some $(n-2)$-face $\sigma_t^{n-2} \subset \sigma_t^n$.

Regge observed the following remarkable variational formula, actually a limiting case of a formula of Schläfli:

$$\left(\sum_{\sigma^{n-2}\subset\partial\sigma^n} \theta(\sigma^{n-2}) \cdot |\sigma^{n-2}|\right)' = \sum_{\sigma^{n-2}} \theta(\sigma^{n-2}) \cdot |\sigma^{n-2}|'\,.$$

That is, the terms involving the derivatives of the dihedral angles cancel one another; their sum vanishes.

Generalized Regge Lemma.

Thus, for a 1-parameter family of piecewise flat metrics X^n_t, on a triangulated manifold with boundary:

$$\big(R^2(X^n_t) + H^2(X^n_t)\big)' = \sum_{\sigma^{n-2}} P_\chi(C^\perp(\sigma^{n-2})) \cdot |\sigma^{n-2}|'\,.$$

More generally, [CMS, 1984], for all Lipschitz-Killing curvatures:

$$(R(X^n_t) + H(X^n_t))' = \sum_{\sigma^{n-2i}\subset X^n} P_\chi(C^\perp(\sigma^{n-2i})) \cdot |\sigma^{n-2i}|'\,.$$

Note, this is consistent with Chern-Gauss-Bonnet formula:

$$(R(X^n_t) + H(X^n_t))' = \chi(X^n_t)' = 0\,.$$

The piecewise flat approximation scheme.

Let M^n be equipped with a *sufficiently fine*, Θ-*fat*, smooth triangulation, T_η, whose edges are *geodesic segments* of length $\sim \eta$.

Here, Θ-*fat* means that for each i-simplex, σ^i,

$$|\sigma^i| \geq \Theta \cdot \eta^i \,.$$

For $\Theta = \Theta(n) > 0$ sufficiently small, such T_η exist for all sufficiently small $\eta > 0$.

Moreover, there is a piecewise flat space X^n_η with same combinatorial structure and the same edge lengths as T_η.

Below, we denote an i-simplex of X^n_η by σ^i_η.

Main theorem; [CMS, 1984].

Let $|R|$, $|\nabla R|$ denote respectively, the norm of the curvature tensor of M^n and of its covariant derivative.

Let $U^n \subset M^n$ be a submanifold with smooth boundary.

Let $B_r(\partial U^n)$ denote the r-tubular neighborhood of ∂U^n.

$$R^i(U^n) := \int_{U^n} R^i \,,$$

$$R^i_\eta(U^n) := \sum_{\sigma^{n-i} \subset U} P_\chi(C^\perp(\sigma^{n-i})) \cdot |\sigma^{n-i}| \,.$$

Theorem. There exists $c = c(|R|, |\nabla R|, \Theta)$ such that

$$|R^i(U^n) - R^i_\eta(U^n)| \leq c \cdot \left(\mathrm{Vol}(U^n) \cdot \eta^{1/2} + \mathrm{Vol}(B_{\eta^{1/2}}(\partial U^n))\right) \,.$$

Issue 1. Dihedral angles not quite well defined.

The case $n = 2$ is elementary, but $n > 2$ is not.

1) As we did for $n = 2$, start by expressing (modulo a negligible error) the "angle defects" which enter into $P_\chi(C^\perp(\sigma^{n-i}))$ in terms of differences in dihedral angles for corresponding pairs $\sigma^j \subset \sigma^k$ and $\sigma^j_\eta \subset \sigma^k_\eta$.

Note that for any pair $\sigma^j_\eta \subset \sigma^k_\eta \subset M^n$, and any vertex $\sigma^0_\eta \subset \sigma^j_\eta$, the corresponding dihedral angle depends only on the 1-skeleton of T_η.

But the values obtained from different choices of $\sigma^0_\eta \subset \sigma^j_\eta$ differ by a *nonnegligible* amount.

This must be appropriately taken into account.

Issue 2: Boundedness of $R^i_\eta(U^n)$ as $\eta \to 0$.

2) For $i > 2$, it is not even clear that $R^i_\eta(U)$ remains bounded as $\eta \to 0$.

$R^i_\eta(U^n)$ is a sum of $\sim \eta^{-n}$ contributions, one for each each vertex of T_η which is contained in U^n.

If each contribution were $O(\eta^n)$, boundedness would follow.

For $i > 2$, this is certainly not obvious since, by local riemannian geometry, each contribution can be expressed in terms of "*face angle defects*", and these are only $O(\eta^2)$.

The required estimate, $O(\eta^n)$, obtained via the generalized Regge lemma, is a sort of "miraculous cancellation".

Issue 3: invariance of the curvature polynomial.

3) The resulting expression for $R^i_\eta(U^n)$ looks like a riemann sum approximating the integral over U^n of a polynomial expression in curvature of degree $i/2$.

A priori, this polynomial appears to depend on the choice of triangulation T_η.

So it is not at all obvious that it is equal in the limit as $\eta \to 0$, to the integral of the *invariant* polynomial in curvature corresponding to $R^i(M^n)$?

That this turns out to hold, reflects an averaging effect which remains a bit mysterious.

The key point: invariance.

It suffices to show that as $\eta \to 0$, modulo a negligible error, Gilkey's conditions characterizing $R^i(M^n)$ hold for the polynomial in curvature described on the previous slide.

The nonroutine point is *invariance under orthogonal change of basis.*

Because of the piecewise flat Chern-Gauss-Bonnet formula, the simplest case is that of $R^n(M^n)$.

We will describe the proof in this case; required modifications in the general case are essentially technical.

Triangulations restricted to $B_{\sqrt{\eta}}(p)$.

It suffices to consider $U^n = B_{\sqrt{\eta}}(p)$, a geodesically convex metric ball.

In particular, the exponential map $\exp_p : B_{\sqrt{\eta}}(0) \to B_{\sqrt{\eta}}(p)$ is a diffeomorphism.

Denote by $T_\eta(B_{\sqrt{\eta}}(p))$, the subcomplex consisting of those n-simplices of T_η which are contained in $B_{\sqrt{\eta}}(p)$.

It is a combinatorial ball.

Lemma on independence of triangulation.

Let $R^n(T_\eta(B_{\sqrt{\eta}}(p)))$ denote the degree $n/2$ polynomial in the curvature tensor at p, obtained from the subcomplex of X^n_η corresponding to $T_\eta(B_{\sqrt{\eta}}(p))$.

It is a sum of $\sim \eta^{-n/2}$ monomials of degree n in components of the curvature tensor at p, one for each interior vertex of $T_\eta(B_{\sqrt{\eta}}(p))$.

Each such monomial has size $\sim \eta^{n/2}$.

Lemma. Up to an error of size $\leq c\sqrt{\eta}$, the polynomial $R^n(T_\eta((B_{\sqrt{\eta}}(p)))$ is independent of the triangulation T_η.

Independence implies invariance of $R(B_{\sqrt{\eta}}(p))$.

Corollary. $\lim_{\eta\to 0} R^n(T_\eta((B_{\sqrt{\eta}}(p)))$ is invariant.

Proof. Since T_η has geodesic edges, it's 1-skeleton Σ^1_η is determined by its 0-skeleton Σ^0_η.

Let $O(M^n_p)$ denote the orthogonal group of the tangent space and $\alpha \in O(M^n_p)$.

Then

$$\Sigma^0_\eta \to \exp_p \circ \alpha \circ \exp_p^{-1}(\Sigma^0_\eta)\,.$$

induces an action of $O(M^n_p)$ on $\{\Sigma^1_\eta\}$, and hence on the associated curvature polynomials $R^n(T_\eta(B_{\sqrt{\eta}}(p)))$.

Thus, the independence lemma implies the corollary.

The extension lemma.

The independence lemma is a consequence of the following extension lemma whose proof, via a general position argument, we omit.

Let the triangulations $T_{1,\eta}$, $T_{2,\eta}$ satisfy our conditions.

Lemma. There exist extensions of $T_{1,\eta}(B_{\sqrt{\eta}}(p))$, $T_{2,\eta}(B_{\sqrt{\eta}}(p))$, to triangulations, $\tilde{T}_{1,\eta}$, $\tilde{T}_{2,\eta}$, satisfying our conditions, such that:

The triangulations, $\tilde{T}_{1,\eta}(B_{\sqrt{\eta}+4\eta}(p))$, $\tilde{T}_{2,\eta}(B_{\sqrt{\eta}+4\eta}(p))$, agree in a neighborhood of their common boundary.

Proof of independence.

Since $\tilde{T}_{i,\eta}(B_{\sqrt{\eta}+4\eta}(p))$, $i = 1, 2$, are combinatorial balls,

$$\chi(\tilde{T}_{1,\eta}(B_{\sqrt{\eta}+4\eta}(p))) = \chi(\tilde{T}_{1,\eta}(B_{\sqrt{\eta}+4\eta}(p))) = 1\,.$$

By the piecewise flat Chern-Gauss-Bonnet formula, for $i = 1, 2$,

$$R^n(\tilde{T}_{i,\eta}(B_{\sqrt{\eta}+4\eta}(p))) + H^n(\tilde{T}_{i,\eta}(B_{\sqrt{\eta}+4\eta}(p))) = 1\,.$$

By the lemma on extending triangulations,

$$R^n(\tilde{T}_{i,\eta}(B_{\sqrt{\eta}+4\eta}(p))) = R^n(\tilde{T}_{i,\eta}(B_{\sqrt{\eta}}(p))) + O(\sqrt{\eta})\,.$$

$$H^n(\tilde{T}_{1,\eta}(B_{\sqrt{\eta}+4\eta}(p))) = H^n(\tilde{T}_{2,\eta}(B_{\sqrt{\eta}+4\eta}(p)))\,.$$

These imply the independence lemma.

The Einstein-Schrader tensor.

A piecewise flat metric on a given triangulated manifold is specified by the squares of the edge lengths i.e. by a 1-cochain, with positive coefficients.

Via the Hilbert action principle and Regge's lemma, Schrader defined a Einstein tensor for piecewise flat spaces.

It can be viewed as a 1-cochain with real coefficients.

An appropriate classical limit theorem can be formulated.

It seems likely that it can be proved by an extension of the methods of [CMS, 1984].