Problem Set 1

1. Vector identities

The following identities are often used in electrodynamics.

(i) Vector identities ($a, b, c, d \in \mathbb{R}^3$ vectors):

$$\begin{aligned} a \cdot (b \times c) &= b \cdot (c \times a) = c \cdot (a \times b)\,, \\ a \times (b \times c) &= (a \cdot c)\, b - (a \cdot b)\, c\,, \\ (a \times b) \cdot (c \times d) &= (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)\,. \end{aligned}$$

(ii) Vector field identities ($f : \mathbb{R}^3 \to \mathbb{R}$ scalar field, $v, w : \mathbb{R}^3 \to \mathbb{R}^3$ vector fields):

$$\begin{aligned} \nabla \times (\nabla f) &= 0\,, \\ \nabla \cdot (\nabla \times v) &= 0\,, \\ \nabla \times (\nabla \times v) &= \nabla(\nabla \cdot v) - \Delta v\,, \end{aligned}$$

with $\Delta := \nabla \cdot \nabla$.

(iii) Product rules:

$$\begin{aligned} \nabla \cdot (fv) &= v \cdot (\nabla f) + f(\nabla \cdot v)\,, \\ \nabla \times (fv) &= (\nabla f) \times v + f(\nabla \times v)\,, \\ \nabla \cdot (v \times w) &= w \cdot (\nabla \times v) - v \cdot (\nabla \times w)\,, \\ \nabla(v \cdot w) &= (v \cdot \nabla)w + (w \cdot \nabla)v + v \times (\nabla \times w) + w \times (\nabla \times v)\,. \end{aligned}$$

(iv) Chain rules ($P : \mathbb{R} \to \mathbb{R}^3$ vector field on $\mathbb{R}$):

$$\begin{aligned} \nabla \cdot P(f(x)) &= \dot{P}(f(x)) \cdot \nabla f(x)\,, \\ \nabla \times P(f(x)) &= \nabla f(x) \times \dot{P}(f(x))\,, \end{aligned}$$

with $\dot{P} = \partial P / \partial t$.

(v) Applications of Gauss's and Stokes' theorem ($D \subset \mathbb{R}^3$ domain, ∂D boundary with outer normal n, $S \subset \mathbb{R}^3$ surface with outer normal n, ∂S boundary with unit tangent s):

$$\begin{aligned} \int_D (f\Delta g + \nabla f \cdot \nabla g) \;&=\; \int_{\partial D} f \nabla g \cdot n\,, \\ \int_D (f\Delta g - g\Delta f) \;&=\; \int_{\partial D} (f\nabla g - g\nabla f) \cdot n\,, \\ \int_D \nabla f \;&=\; \int_{\partial D} f\, n\,, \\ \int_S n \times \nabla f \;&=\; \int_{\partial S} f\, s\,. \end{aligned}$$

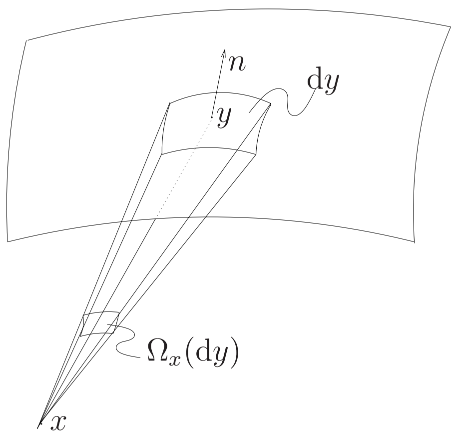
2. Dipole densities

(i) Spatial dipole density $P(y)$: The volume element $\mathrm{d}y$ carries a dipole moment $P(y)\,\mathrm{d}y$. Find the equivalent charge density $\rho(y)$, i.e. the one which generates the same field.

(ii) Areal dipole density $P(y)$: The surface element $n\mathrm{d}y$ carries a dipole moment $P(y)\,n\mathrm{d}y$. Show that the potential of a dipole layer of density $P(y)$ on the surface S is given by

$$\varphi(x) = -\frac{1}{4\pi}\int_S P(y)\,\Omega_x(\mathrm{d}y). \tag{1}$$

Here $\Omega_x(\mathrm{d}y)$ is the solid angle of $n\mathrm{d}y$ seen from x. By convention, $\Omega_x(\mathrm{d}y) > 0$ if n and the line of sight from x to y have a positive scalar product.



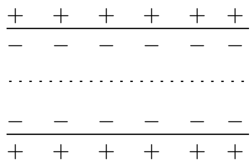
Moreover, show that the potential jumps at the surface:

$$\varphi(x_0 + 0n) - \varphi(x_0 - 0n) = P(x_0)\,, \qquad (x_0 \in S)\,. \tag{2}$$

Hint: For (1), motivate

$$\frac{1}{|y-x|^2}\left(\frac{y-x}{|y-x|}\cdot n\mathrm{d}y\right) = \Omega_x(\mathrm{d}y)\,. \tag{3}$$

Application: Cell membranes are lipid bilayers. Every layer has a positive dipole density on the exterior (see figure). The potential inside the membrane therefore is smaller than the potential outside. The interior of the membrane therefore is energetically more favourable for positively charged ions than for negatively charged ones. While there is no difference for other properties, the permeability of the membrane consequently is higher for the former. The jump of the potential at the single layer is of the order of magnitude of 0.1 V.



3. Homogeneously charged and homogeneously polarized solid sphere

(i) Calculate the electric field and potential of a homogeneously charged solid sphere (charge Q) of radius R.

(ii) Calculate the electric field of a homogeneously polarized solid sphere (density P) of radius R.

Due: 09.03.2016.