Problem Set 2

1. A grounded conducting sphere in a homogeneous external electric field

Let a grounded spherical electric conductor be placed in a homogeneous external field $E_\infty = \lim_{|x|\to\infty} E(x)$. Find the resulting electric field $E(x)$.

Hint: Which field equations and which boundary conditions should be satisfied by the potential? Use the solution of problem set 1, ex. 3(ii).

2. The Cavendish experiment

The Cavendish experiment (1773) is a test of Coulomb's law. Two concentric, conducting, hollow spheres of radii $R_1 < R_2$ are connected by a wire and isolated outwards. Let the charges of the two spheres be Q_1 and Q_2 respectively.

(i) Show $Q_1 = 0$ without much calculation. The spheres may be replaced by arbitrary surfaces, as long as the outer surface is closed.

Replace now Coulomb's law by

$$F_{12} = e_1 e_2 F(r)\frac{x}{r}, \quad (r = |x|),$$

with an arbitrary force law $F(r)$.

(ii) Show that the potential of a homogeneously charged sphere of radius a and charge 1 is

$$V(r;a) = \frac{f(r+a) - f(|r-a|)}{2ar},$$

where

$$f(r) = \int_0^r U(s)s\,ds\,, \qquad \frac{dU}{dr} = -F(r)\,.$$

The potential U is determined up to $U(r) \to U(r) + C$, ($C = \text{const}$). How does this affect f? What is f in the case of Coulomb's law?

(iii) Calculate the ratio of the charges,

$$\frac{Q_1}{Q_2} = \frac{R_1}{R_2} \cdot \frac{f(2R_2)R_1 - (f(R_2+R_1) - f(R_2-R_1))R_2}{f(2R_1)R_2 - (f(R_2+R_1) - f(R_2-R_1))R_1}\,. \tag{1}$$

Moreover, show that $Q_1 = 0$ for all radii only in the case of Coulomb's law.

Hint: In equilibrium, the total potential $\sum_{i=1,2} Q_i V(r;R_i)$ on both spheres $r = R_j$ ($j = 1, 2$) is the same.

3. Thomson's theorem

Let several bodies be fixed, with each of them given a fixed total charge respectively. The electrostatic energy attains a minimum if the charges are distributed in a way such that the potential is constant on each of them (as in the case of conductors). In particular, there is no charge inside the bodies.

Hint: Express the energy in terms of the charge density.

4. Electrostatic energy in an external field

Let a charge density $\rho(x)$ in a neighbourhood of $x = 0$ be given, as well as an external potential $\varphi(x)$. The latter is nearly constant in this neighbourhood of $x = 0$, and its sources are located outside. Show that the electrostatic energy of the charge density in the field $E = -\nabla\varphi$ of the external potential can be expressed as follows:

$$W = e\varphi(0) - p \cdot E(0) - \frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \dots . \tag{2}$$

Here $e = \int \rho(x)\,dx$ is the total charge, $p = \int x\rho(x)\,dx$ is the dipole moment, and $Q_{ij} = \int (3x_i x_j - x^2\delta_{ij})\rho(x)\,dx$ is the quadrupole moment.

Hint: Expand the potential φ in $W = \int \rho(x)\varphi(x)\,dx$ around $x = 0$ using Taylor series.

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