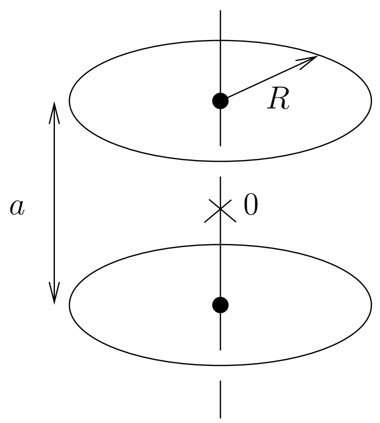
Problem Set 3

1. Helmholtz coil

(i) Let a current I pass through a circular conductor of radius R. What are the symmetries of the magnetic field? Find the field on the symmetry axis of the conductor (w.l.o.g. the 3-axis).

A Helmholtz coil consists of two circular conductors with a common symmetry axis.



Let the two currents be equal, and oriented in the same direction. The set-up produces a magnetic field which is nearly constant within a certain area. More precisely: how should the distance a be chosen in order for $B(x) - B(0) = O(|x|^4)$, $(x \to 0)$, to hold true?

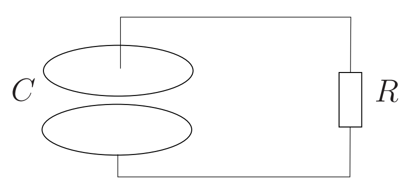
(ii) Show: this happens if
$$\frac{\partial^2 B_3}{\partial x_3^2} = 0 \tag{1}$$
at the center.

Hint: Use the symmetries of the field and the field equations, without determining a.

(iii) Determine a such that (1) holds true.

2. Energy flow due to the discharge of a capacitor

Discuss qualitatively the energy flow (Poynting vector) due to the slow discharge of a capacitor via a resistor.



3. Completely and partially polarized light

A monochromatic wave in propagation direction e_3 is, in complex notation, of the form

$$\begin{aligned} B &= e_3 \times E\,, \qquad & E(x,t) &= E(t - e_3 \cdot x/c)\,, \\ E(t) &= E_0 \mathrm{e}^{-i\omega_0 t}\,, \qquad & E_0 &= (E_1, E_2, 0)\,. \end{aligned} \tag{2}$$

The polarization of the wave is described by the complex vector

$$\underline{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \in \mathbb{C}^2, \tag{3}$$

which contains four real degrees of freedom.

A measurement of polarization first of all filters the wave, namely in the direction of a particular polarization $\underline{\varepsilon}$, $((\underline{\varepsilon}, \underline{\varepsilon}) = 1)$, i.e. $\underline{E} \rightsquigarrow \underline{E}' = (\underline{\varepsilon}, \underline{E})\underline{\varepsilon}$. Subsequently the intensity

$$I = \frac{c}{2}(\underline{E}', \underline{E}') = \frac{c}{2}|(\underline{\varepsilon}, \underline{E})|^2$$

is measured. If the polarization is changed by a phase, $\underline{E} \rightsquigarrow \mathrm{e}^{i\varphi}\underline{E}$, only the physical field $\operatorname{Re} E(t)$ is delayed, which has no influence on the measurements. Thus there remain three degrees of freedom which have to be expressed in an experimentally useful way.

A *quasi*-monochromatic wave is described by the replacement $\underline{E} \rightsquigarrow \underline{E}(t)$ in (3), where the corresponding amplitude $E_0(t)$ in (2) is now

- changing slowly on the time scale $2\pi/\omega_0$ (period); the characteristic time scale of $E_0(t)$ is called coherence time τ.
- changing rapidly on the time scale of the measurements. The amplitudes $E_0(t_i)$, $(i = 1, 2)$, turn out to be uncorrelated if $t_2 - t_1 \gg \tau$.

Regard $\underline{E}$ as a random variable. We denote mean values by $\langle \cdot \rangle$. The strictly monochromatic wave corresponds to the deterministic special case of a pure polarization.

The goal of the exercise is to understand the following statement: The polarization of light is described by the matrix

$$S = \begin{pmatrix} \langle E_1 \overline{E}_1 \rangle & \langle E_1 \overline{E}_2 \rangle \\ \langle E_2 \overline{E}_1 \rangle & \langle E_2 \overline{E}_2 \rangle \end{pmatrix} = S^*\,, \qquad \text{i.e. } S = \langle \underline{E}\, \underline{E}^* \rangle, \tag{4}$$

where $\underline{E}^* = (\overline{E}_1, \overline{E}_2)$.

(i) Show that S is of the form

$$S = s_0\sigma_0 + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 \equiv s_0 \mathbb{1}_2 + s \cdot \sigma, \tag{5}$$

where $s_i \in \mathbb{R}$ (four stokes parameters), $\sigma_0 = \mathbb{1}_2$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Pauli matrices).

Hint: The 2×2-matrices $S = S^*$ form a real vector space. What is its dimension?

(ii) Express s_i using the mean values $\langle E_j \overline{E}_k \rangle$.

Hint: $(S, T) = \mathrm{tr}(ST)/2$ is a scalar product. Furthermore $\sigma_i^2 = \mathbb{1}_2$ and $\sigma_1\sigma_2 = i\sigma_3$ (the latter also for cyclic permutations of the indices).

(iii) Show that in the case of a pure polarization (E_0 fixed, mean values not necessary)

$$|s| = s_0. \tag{6}$$

This corresponds to the mentioned three degrees of freedom.

Hint: Compute S^2 in this case an take the trace.

(iv) Show that generally

$$|s| \leqslant s_0 \, .$$

Hint: Average over (6); alternatively show: $S \geqslant 0$ and the eigenvalues of S are $s_0 \pm |s|$.

(v) Find the meaning of s_0 and s_i/s_0, $(i = 1, 2, 3)$. What does $s = 0$ mean?

Hint: The eigenvalues of σ_i are ± 1, $(i = 1, 2, 3)$. What are the eigenvectors $e_\pm^{(i)}$? Express s_i using the coefficients $\alpha_\pm^{(i)}$ of the decomposition $\underline{E} = \alpha_+^{(i)} e_+^{(i)} + \alpha_-^{(i)} e_-^{(i)}$. Use that the trace does not depend on the basis.

Remark: In optics the matrices σ_i are defined a bit differently. Here the Pauli matrices, which are common in quantum mechanics, are used.

Due: 23.03.2016.