Problem Set 4

1. Solution of the inhomogeneous wave equation

In class we saw how to solve the homogeneous wave equation $\Box u = 0$ on $\mathbb{R} \times \mathbb{R}^3$ using the Fourier transform. Here $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$ is the d'Alambert operator.

(i) Using the Fourier transform, solve the inhomogeneous wave equation $\Box u = f$ with given initial data $u(0,\cdot)$ and $\partial_t u(0,\cdot)$. Here $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a source.

(ii) Suppose that $u(0,\cdot) = \partial_t u(0,\cdot) = 0$ and that $f(s,x) = 0$ for $s \leqslant 0$. Show that your solution from (i) coincides with the retarded solution of the inhomogeneous wave equation derived in class.

Hints: Use the Fourier transform

$$\hat{u}(t,k) \;:=\; \int_{\mathbb{R}^3} \mathrm{d}x\, u(t,x)\, \mathrm{e}^{-\mathrm{i}x\cdot k}\,, \qquad u(t,x) \;=\; \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{(2\pi)^3}\, \hat{u}(t,k)\, \mathrm{e}^{\mathrm{i}x\cdot k}\,.$$

To solve the resulting ordinary differential equation, recall *Duhamel's principle*, or the *variation of constants formula*, which states that the solution of the differential equation $X'(t) = AX(t) + F(t)$ is $X(t) = \mathrm{e}^{At}X(0) + \int_0^t \mathrm{d}s\, \mathrm{e}^{A(t-s)}F(s)$, for any vector-valued function X and square matrix A.

You will also need to use (and prove!) the identity

$$\frac{\sin(|k|r)}{|k|r} \;=\; \frac{1}{4\pi r^2}\int_{\partial B_r(0)} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}k\cdot y} \tag{1}$$

for $r > 0$, stated in class.

2. Solution of the wave equation in two dimensions

(i) Find the distributional solution $D_2(t,\underline{x})$ for the initial value problem

$$\begin{aligned} \Box u(t,\underline{x}) = 0, \qquad & (t,\underline{x}) \in \mathbb{R}\times\mathbb{R}^2\,, \\ u(0,\underline{x}),\ \partial_t u(0,\underline{x}) \qquad & \text{given} \end{aligned}$$

in dimension 2.

Hint: Expand the formulation of the exercise to a 3-dimensional problem, with $(t,\underline{x},x_3) \in \mathbb{R}\times\mathbb{R}^3$, which is invariant under translations w.r.t. x_3 and compute the 2-dimensional distributional solution from the 3-dimensional one.

(ii) In 3 dimensions, $u(t,x)$ is uniquely determined by the values of $u(\tilde{t},\tilde{x})$, with $(x-\tilde{x})^2 = c^2(t-\tilde{t})^2$ (light cone). What changes in the 2-dimensional case?

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