Problem Set 6

1. Applications of Lorentz transformations

(i) *Time dilation.* The two events A and B occur at the same place in an inertial frame O (e.g. time measurements w.r.t. a clock which is at rest in O). Using boosts, show that the time difference in an inertial frame O', which is moving w.r.t. O, is larger.

(ii) *Length contraction.* Consider a rigid rod which has length L in its rest frame O. Show that the length of the rod is smaller in an inertial frame O' moving longitudinal to it.

Hint: The length is given by the coordinate difference of the end points of the rod at the same time.

(iii) Consider two inertial systems O and O' with parallel axes, where O' is moving in 1-direction with relative velocity v w.r.t. O. The rod is again pointing in 1-direction, but now is moving with velocity w in 2-direction. Determine its angle θ to the 1-direction w.r.t. O'.

(iv) Let $g = \mathrm{diag}(1, -1, -1, -1)$, and $\langle \xi, \xi \rangle := \xi^\top g \xi$. A four-vector ξ is called *timelike* if $\langle \xi, \xi \rangle > 0$, and *spacelike* if $\langle \xi, \xi \rangle < 0$. Show: two events x, y are at the same time in an appropriate inertial system if and only if $x - y$ is spacelike. They occur at the same place in an appropriate inertial system if and only if $x - y$ is timelike.

2. Lorentz transformations on the celestial sphere

The celestial sphere is an imaginary sphere of arbitrarily large radius, concentric with Earth. All objects in the observer's sky can be thought of as projected onto this sphere. More precisely, the position of stars on it is given by the direction their light comes from. If two inertial systems are rotated or moved w.r.t. each other, the light of a star comes from different directions w.r.t. the two systems (aberration). Consequently, the night sky looks different for the two observers. The goal of this exercise is to show that the transformation from one to the other is Möbius.

A transformation on the extended complex plane $\mathbb{C} \cup \{\infty\}$ is called Möbius if it is of the form

$$z \longmapsto \frac{az + b}{cz + d}, \qquad (a, b, c, d \in \mathbb{C}),$$

with $ad - bc \neq 0$. The composition of Möbius transformations corresponds to the product of the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Two matrices A, B define the same transformation if and only if $B = \lambda A$, $\lambda \neq 0$. After normalizing $\det A = 1$, we are left with the choice $\lambda = \pm 1$. Therefore the Möbius group is $\mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$.

The plane $\mathbb{C} \cup \{\infty\}$ can be regarded as the Riemann sphere $S^2 = \{\underline{v} \in \mathbb{R}^3 \,|\, |\underline{v}| = 1\}$ using stereographic projection. In both descriptions (plane and sphere), a transformation is Möbius if and only if it preserves circles and the orientation.

Show that if S^2 is considered as the celestial sphere of the directions of light, then a Lorentz transformation $\Lambda \in L_+^\uparrow$ induces a Möbius transformation $\pm A$ on S^2. For this purpose, note that Lorentz transformations are characterized by the following properties:

(i) Invariance of form of the law of inertia

$$\underline{x} = \underline{b} + \underline{v}t \qquad \Longleftrightarrow \qquad \underline{x}' = \underline{b}' + \underline{v}'t' \,.$$

(ii) Invariance of the speed of light ($c = 1$)

$$|\underline{v}| = 1 \qquad \Longleftrightarrow \qquad |\underline{v}'| = 1 \,.$$

Consider the map $S : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\underline{v} \mapsto \underline{v}'$, which is induced by Λ. Using (i), show that S is collinear, even though S is not linear as opposed to $\Lambda : \mathbb{R}^4 \mapsto \mathbb{R}^4$.

Hint: What does "the points $\underline{v}_1, \underline{v}_2, \underline{v}_3$ lie on a line in $\mathbb{R}^3$" mean in terms of inertial trajectories in $\mathbb{R}^{1+3}$?

Using (ii), conclude that S maps circles in S^2 onto circles in S^2.

Hint: What is such a circle w.r.t $S^2 \subset \mathbb{R}^3$?

The result can be summarized in the isomorphism

$$L_+^\uparrow \cong \mathrm{SL}(2, \mathbb{C})/\{\pm 1\} \,.$$

3. Doppler shift and aberration

(i) Let the field $\varphi(\underline{x}, t)$ be a scalar field under Lorentz transformations $x' = \Lambda x$, i.e. $\varphi'(x') = \varphi(\Lambda^{-1}x')$. Show that for a wave $\varphi(\underline{x}, t) = e^{\imath(\underline{k}\cdot\underline{x} - \omega t)}$, the frequency and the wave vector form a 4-vector $k \equiv (\omega/c, \underline{k})$.

Henceforth let O and O' be connected via a boost $\Lambda = \Lambda(v\underline{e}_1)$ in 1-direction. Consider light with direction of propagation $\underline{e}$ and frequency ω w.r.t. O.

(ii) Let $\underline{e} = \underline{e}_1$. Compute the frequency w.r.t. O' (*Doppler shift*).

(iii) Let $\underline{e} = \underline{e}_2$. Determine the angle $\alpha(v)$ between the wave vector and the 2-axis w.r.t. O' (*Aberration*).

4. Transformation of velocities

Let two inertial systems O and O' with parallel axes be given. O' is moving with a relative velocity v w.r.t. O in 1-direction. A particle has velocity $\underline{w} = (w_1, w_2, w_3)$ w.r.t. O. Compute the components of the velocity w.r.t. O'.

Hint: Express the velocity in terms of the four-velocity.

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