Problem Set 8

1. Calculating with commutators

(i) The commutator [A, B] = AB - BA of two matrices or operators A, B is linear in A, B and antisymmetric: [B, A] = -[A, B]. Show that the product rule

$$[A, BC] = [A, B]C + B[A, C]$$

and the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

hold true.

(ii) Using $(i/\hbar)[P, X] = 1$, show that

$$\frac{\mathrm{i}}{\hbar}[P, f(X)] = f'(X) , \qquad \frac{\mathrm{i}}{\hbar}[g(P), X] = g'(P)$$

for polynomials f(x), g(p), where f(X), g(P) are defined through sums and products of matrices.

(iii) Derive the commutation relations for the angular momentum L: Show that the components $L_i = X_{i+1}P_{i+2} - X_{i+2}P_{i+1}$ $(i = 1, 2, 3 \mod 3)$ satisfy the commutation relations

$$[L_{i+1}, L_{i+2}] = \mathrm{i}\hbar L_i \,.$$

Hint: $(i/\hbar)[P_i, X_j] = \delta_{ij}$.

Show that the expectation value of the components L_1 and L_2 vanishes in every eigenstate of L_3 .

2. Moving on a line and tunnelling

A classical particle needs to have a kinetic energy which is as least as big as the potential it has to overcome. This is not the case in quantum mechanics, as shall be seen in the following exercise.

A particle of energy E travels from $x = -\infty$ towards the potential

$$V(x) = \begin{cases} 0 , & (|x| \ge a/2) \\ V_0 , & (|x| < a/2) \end{cases} \quad (a > 0, V_0 > 0) .$$



Solve the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}(x) + V(x)\psi(x) = E\psi(x)$$
(1)

with the ansatz

$$\psi_{I}(x) = A_{1}e^{ikx} + B_{1}e^{-ikx} \qquad (x < -a/2) , \qquad k = k(E)$$

$$\psi_{II}(x) = A_{2}e^{ilx} + B_{2}e^{-ilx} \qquad (|x| < a/2) , \qquad l = l(E, V_{0})$$

$$\psi_{III}(x) = A_{3}e^{ikx} \qquad (x > a/2) .$$

The parts $A_1 e^{ikx}$, $A_3 e^{ikx}$ and $B_1 e^{-ikx}$ describe an incident (in), a transmitted (t), and a reflected (r) wave (resp. particle) respectively. The transmission and reflection coefficients are given by

$$T = \frac{j_{\rm t}}{j_{\rm in}}$$
, $R = \frac{|j_{\rm r}|}{j_{\rm in}}$

where the j are the corresponding (constant) current densities. Compute T = T(E) and R = R(E) for the cases (a) $0 < E < V_0$ and (b) $0 < V_0 < E$, with the result: R + T = 1 and

$$T(E) = \frac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sinh^2\left(\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}a\right)}$$

in case (a); and with the replacements $V_0 - E \rightsquigarrow E - V_0$, sinh \rightsquigarrow sin in case (b).

Hints: • Reason why ψ and $d\psi/dx$ are continuous even at the discontinuities $x = \pm a/2$ of the potential. • In case (a), l is purely imaginary. • Intermediate result: $T = |A_3/A_1|^2$ and $R = |B_1/A_1|^2$.

Sketch the development of T(E). Classically, we would have (a) T(E) = 0, (b) T(E) = 1. In contrast to that, the quantum mechanical result is (a) T(E) > 0 (tunnelling) and (b) $T(E_n) = 1$ only for specific energies E_n (transmission resonances). For which? What happens for $\hbar \to 0$?

3. Current and momentum

Let $\hbar = 2m = 1$. A particle on a line has

$$j(x) = 2 \operatorname{Im}\overline{\psi(x)}\psi'(x) \tag{2}$$

as the expectation value of the current at x. This is a local property of $\psi(x)$, as opposed to the expectation of the momentum p = k.

Show the following, a bit surprising fact: there are superpositions

$$\psi(x) = a_1 e^{ik_1x} + a_2 e^{ik_2x}, \quad (k_1, k_2 > 0)$$

of waves traveling to the right, for which the current at x = 0 (at the considered time) travels to the left: j(0) < 0.

Remark (beyond this exercise): The superposition is not normalizable, but there are analog wave packages which indeed are normed.

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