Problem Set 11

1. Half a harmonic oscillator

Consider a particle in one dimension and in the potential

$$V(x) = \begin{cases} \infty\,, & (x<0)\,, \\ \frac{1}{2}x^2\,, & (x>0)\,. \end{cases}$$

Find the eigenvalues and eigenfunctions of the corresponding Hamiltonian in units $m = \hbar = 1$.

Hint: The divergent part of the potential can be replaced by a boundary condition $\psi(x = 0) = 0$ for the wave function defined on $x \geq 0$, cf. exercise 9.1.

2. Time-energy uncertainty principle

For an operator A and a state ψ_t define $(\Delta A)^2 := \mathrm{Var}_{\psi_t}(A)$. The position-momentum uncertainty principle can then be written $\Delta p \cdot \Delta x \geqslant \hbar/2$. In literature, one finds the claim

$$\Delta H \cdot \Delta t \geqslant \frac{\hbar}{2}\,, \tag{1}$$

H the Hamiltonian, which seems to bear some similarity to the position-momentum uncertainty principle. However, the interpretation of (1) is complicated by the fact that there is no "time-observable" in quantum mechanics. Below are two possible interpretations (both by Mandelshtam, Tamm 1945).

(i) The expectation value of an observable A changes at a rate $\dot{A} := d\langle A\rangle_{\psi_t}/dt$. The time t (according to this interpretation) is the one at which A exceeds or falls below a specific value. Since the measurement of A is subject to a variation ΔA, we define

$$\Delta t := \frac{\Delta A}{|\dot{A}|}\,.$$

Show (1).

Hints: Use the equation of motion in the Heisenberg picture,

$$\dot{A}(t) = \frac{\mathrm{i}}{\hbar}[H, A(t)]\,. \tag{2}$$

(ii) The evolution of a state ψ_0 is given by the Schrödinger equation

$$\mathrm{i}\hbar\frac{d}{dt}\psi_t = H\psi_t\,. \tag{3}$$

Let $t_0 > 0$ be a time at which the state ψ_t is definitely distinguishable from the initial state ψ_0:

$$\langle\psi_0, \psi_{t_0}\rangle = 0\,.$$

Show:

$$\Delta H \cdot t_0 \geqslant \hbar \cdot \frac{\pi}{2}\,.$$

Hints: Estimate $|\dot{f}(t)|$, where $f(t) = |\langle \psi_0, \psi_t \rangle|^2$. The computation yields the expression $\langle \psi_t, [P, H]\psi_t \rangle$ with $P = \psi_0 \psi_0^*$; use then the uncertainty principle. Eventually, use the comparison

$$\dot{f}(t) \geqslant g(f(t)),\ \dot{f}_0(t) = g(f_0(t)),\ f(0) = f_0(0) \quad \Longrightarrow \quad f(t) \geqslant f_0(t),\ (t \geqslant 0)\,. \tag{4}$$

(iii) Conversely, interpretation (ii) has a counterpart for position and momentum, e.g. in dimension 1: let ψ_x be the state ψ_0 shifted by x, and let x_0 be such that $\langle \psi_0, \psi_{x_0} \rangle = 0$. Show:

$$\Delta p \cdot x_0 \geqslant \hbar \cdot \frac{\pi}{2}\,.$$

Hint: Choose H appropriately in (3).

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