Solution 1

1. Vector identities

Remark: We use the following conventions below: x_i denotes the i-th component of the vector x, δ_{ij} is the Kronecker delta, and repetition of an index variable implies summation over all the values of the index, e.g. $x_i y_i \equiv \sum_i x_i y_i$.

(i) The first identity follows from $a \cdot (b \times c) = \varepsilon_{ijk} a_i b_j c_k$ and ε_{ijk} being invariant under cyclic permutations of the indices. The second and third identity follow from

$$(a \times (b \times c))_i = \varepsilon_{ijk}\varepsilon_{klm} a_j b_l c_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m \,,$$
$$(a \times b) \cdot (c \times d) = (\varepsilon_{ijk} a_j b_k)(\varepsilon_{ilm} c_l d_m) = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k c_l d_m \,.$$

(ii) We write $\partial_i = \partial/\partial x^i$,

$$(\nabla \times (\nabla f))_i = \varepsilon_{ijk}\partial_j \partial_k f = 0 \,.$$
$$\nabla \cdot (\nabla \times v) = \varepsilon_{ijk}\partial_i\partial_j v_k = 0 \,.$$
$$(\nabla \times (\nabla \times v))_i = \varepsilon_{ijk}\varepsilon_{klm}\partial_j\partial_l v_m = \partial_j\partial_i v_j - \partial_j\partial_j v_i \,.$$

(iii)

$$\nabla \cdot (fv) = \partial_i (f v_i) = (\partial_i f) v_i + f \partial_i v_i \,.$$
$$(\nabla \times (fv))_i = \varepsilon_{ijk}\partial_j (f v_k) = \varepsilon_{ijk}(\partial_j f) v_k + f\varepsilon_{ijk}\partial_j v_k \,.$$
$$\nabla \cdot (v \times w) = \varepsilon_{ijk}\partial_i (v_j w_k) = \varepsilon_{ijk}(\partial_i v_j) w_k - \varepsilon_{jik} v_j \partial_i w_k \,.$$

For the fourth identity we first calculate

$$\begin{aligned}((v \cdot \nabla) w + v \times (\nabla \times w))_i &= (v_j \partial_j) w_i + \varepsilon_{ijk} v_j (\varepsilon_{klm}\partial_l w_m) \\ &= (v_j\partial_j) w_i + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j \partial_l w_m = v_j \partial_i w_j \,.\end{aligned}$$

Exchanging v and w and adding the two expressions leads to the claimed result.

(iv)

$$\nabla \cdot P(f(x)) = \partial_i P_i(f(x)) = \dot{P}_i(f(x))\partial_i f(x) \,.$$
$$(\nabla \times P(f(x)))_i = \varepsilon_{ijk}\partial_j P_k(f(x)) = \varepsilon_{ijk}\dot{P}_k(f(x))\partial_j f(x) \,.$$

(v) Let $e^{(i)}$, $(i = 1, 2, 3)$ be a fixed basis vector.

$$\int_{\partial D} f \nabla g \cdot n \overset{\text{Gauss}}{=} \int_D \nabla \cdot (f \nabla g) = \int_D (\nabla g \cdot \nabla f + f(\nabla \cdot \nabla g)) \,.$$
$$\int_D (f \Delta g - g \Delta f) = \int_{\partial D} (f \nabla g - g \nabla f) \cdot n - \int_D (\nabla g \cdot \nabla f - \nabla f \cdot \nabla g) \,.$$
$$\int_D \nabla f \cdot e^{(i)} = \int_{\partial D} f e^{(i)} \cdot n \,, \quad (\nabla g = e^{(i)} \text{ in the first identity}) \,.$$
$$\int_{\partial S} f e^{(i)} \cdot s \overset{\text{Stokes}}{=} \int_S \nabla \times f e^{(i)} \cdot n = \int_S (\nabla f \times e^{(i)} + f(\nabla \times e^{(i)})) \cdot n \,.$$

2. Dipole densities

(i) The charge distribution corresponding to $P(y)\,\mathrm{d}y$ is $-P(y)\,\nabla_x\delta(x-y)\,\mathrm{d}y$ by (1.15), thus using $\nabla_x f(x-y) = -\nabla_y f(x-y)$ we get

$$\begin{aligned}\rho(x) &= \int P(y)\cdot\nabla_y\delta(x-y)\,\mathrm{d}y\\ &= -\int(\nabla\cdot P(y))\delta(x-y)\,\mathrm{d}y\\ &= -\nabla\cdot P(x)\,.\end{aligned} \tag{4}$$

Alternative version: by (1.16) the potential is given by

$$\begin{aligned}\varphi(x) &= -\int P(y)\cdot\nabla_x\frac{1}{4\pi|x-y|}\mathrm{d}y\\ &= -\frac{1}{4\pi}\int\frac{\nabla\cdot P(y)}{|x-y|}\,\mathrm{d}y\,,\end{aligned} \tag{5}$$

which is the potential corresponding to charge density (4).

(ii) Here, analog to (5),

$$\begin{aligned}\varphi(x) &= \int_S P(y)\nabla_y\frac{1}{4\pi|x-y|}\cdot n\mathrm{d}y\\ &= -\frac{1}{4\pi}\int_S P(y)\frac{y-x}{|y-x|^3}\cdot n\mathrm{d}y\,.\end{aligned} \tag{6}$$

Furthermore

$$\frac{1}{|y-x|^2}\Big(\frac{y-x}{|y-x|}\cdot n\mathrm{d}y\Big) = \Omega_x(\mathrm{d}y)\,,$$

since the scalar product projects n onto the direction of the line of sight from x to y. Hence (1) follows.

Consider a small environment of $x_0\in S$ in S, inside which $P(y)$ is constant up to a small error. For $x\to x_0\pm 0n$, this environment has a solid angle $\mp 2\pi$ as seen from x. For the other points in S, $\Omega_x(\mathrm{d}y)\to\Omega_{x_0}(\mathrm{d}y)$ independent of $\pm$. Therefore

$$\varphi(x_0+0n)-\varphi(x_0-0n) = P(x_0)\,.$$

Alternative considerations:

(a) The dipole layer consists of two surfaces which are separated from each other by Dn, with areal charge density $\pm\sigma$, in the limit $D\to 0$, $\sigma\to\infty$, $\sigma D\to P$. In between the surfaces the field $E=-\sigma n$ dominates to leading order, therefore the difference of the potentials is $-E\cdot Dn\to P$.

(b) The potential (6) of the dipole layer,

$$\varphi(x) = -\sum_{i=1}^{3}\int P(y)n_i\frac{\partial}{\partial x_i}\frac{1}{4\pi|x-y|}\mathrm{d}y\,,$$

and the electric field of a charge layer,

$$E_i(x) = -\int\sigma(y)\frac{\partial}{\partial x_i}\frac{1}{4\pi|x-y|}\mathrm{d}y\,,$$

exhibit a formal analogy. And like the latter has a jump $E_i|_1^2 = \sigma n_i$, the former has one $\varphi|_1^2 = \sum_{i=1}^3 Pn_i^2 = P$.

3. Homogeneously charged and homogeneously polarized solid sphere

(i) The charge inside $|x| \leqslant r$ is

$$Q(r) = \begin{cases} Q\,, & (r \geqslant R) \\ Q(\frac{r}{R})^3\,, & (r \leqslant R)\,. \end{cases}$$

By Gauss's law (1.7), and by taking into account the symmetry (E radial), we have

$$E(x) = \begin{cases} \frac{1}{4\pi} Q \frac{x}{|x|^3}\,, & (|x| > R) \\ \frac{1}{4\pi} \frac{Q}{R^3} x\,, & (|x| \leqslant R)\,. \end{cases} \tag{7}$$

Therefore the potential is $\varphi(r) = -\int_\infty^r |E|(r')\mathrm{d}r'$, ($r = |x|$, $|E|(r) = |E(x)|$)

$$\varphi(r) = \begin{cases} -\frac{Q}{4\pi} \int_\infty^r \frac{\mathrm{d}r'}{r'^2} = \frac{Q}{4\pi r}\,, & (r \geqslant R) \\ -\frac{Q}{4\pi} \big(\int_\infty^R \frac{\mathrm{d}r'}{r'^2} + \int_R^r r'\,\mathrm{d}r' \big) = \frac{Q}{4\pi R} \big(\frac{3}{2} - \frac{1}{2}(\frac{r}{R})^2 \big)\,, & (r < R). \end{cases}$$

(ii) By (5) the potential is

$$\varphi(x) = -P \cdot \nabla_x \int_{|y| \leqslant R} \frac{1}{4\pi |x-y|} \mathrm{d}y\,.$$

The integral is recognized to be the potential of a solid sphere of charge density 1 (charge $Q = 4\pi R^3/3$). Thus

$$\varphi(x) = P \cdot E_a(x) = \begin{cases} R^3 \frac{P \cdot x}{3|x|^3}\,, & (|x| > R) \\ \frac{P \cdot x}{3}\,, & (|x| \leqslant R)\,, \end{cases} \tag{8}$$

where E_a is the field (7) from part (a). For the electric field one finds

$$E(x) = \begin{cases} R^3 \frac{3(P \cdot x)x - Px^2}{3|x|^5}\,, & (|x| > R) \\ -\frac{P}{3}\,, & (|x| \leqslant R)\,. \end{cases} \tag{9}$$