Solution 2

1. A grounded conducting sphere in a homogeneous external electric field

Let R be the radius of the sphere. The potential satisfies

$$\begin{aligned}\varphi(x) &= 0, \quad (|x| \leqslant R),\\ \Delta\varphi(x) &= 0, \quad (|x| > R),\\ \varphi(x) &= -E_\infty \cdot x + O(1), \quad (|x| \to \infty)\,.\end{aligned}$$

Subtracting the contribution of the homogeneous field, $\tilde{\varphi}(x) = \varphi(x) + E_\infty \cdot x$, we get

$$\begin{aligned}\tilde{\varphi}(x) &= E_\infty \cdot x, \quad (|x| \leqslant R),\\ \Delta\tilde{\varphi}(x) &= 0, \quad (|x| > R),\\ \tilde{\varphi}(x) &= O(1), \quad (|x| \to \infty)\,.\end{aligned}$$

A solution (and indeed the only one) thereof is eq. (8) in the solution of problem set 1, with $P = 3E_\infty$. Thus, by the subsequent equation (9), we have

$$E(x) = \begin{cases} R^3 \frac{3(E_\infty \cdot x)x - E_\infty x^2}{|x|^5} + E_\infty\,, & (|x| > R)\\ 0\,, & (|x| \leqslant R)\,. \end{cases}$$

2. The Cavendish experiment

(i) Let S_1 and S_2 be the inner and the outer surface respectively. The potential φ is constant across both surfaces, since they are connected. Moreover, as S_2 is closed, φ is constant everywhere inside S_2, since this is the solution of $\Delta\varphi = 0$ for the boundary conditions given. The charge density on S_1 is the jump $-\nabla\varphi \cdot n$ and therefore vanishes. (S_2 is a Faraday cage.)

(ii) $U(s)$ is the potential of a point charge with charge 1 at 0. The distance s from x to some point on the sphere with angle θ with respect to x is given by $s^2 = r^2 + a^2 - 2ar\cos\theta$. Such points have a surface charge $2\pi \sin\theta d\theta/4\pi$. Therefore

$$V(r) = \frac{1}{2}\int_0^\pi d\theta \sin\theta\, U\big(\sqrt{r^2 + a^2 - 2ar\cos\theta}\big).$$

The substitution $\theta \to s$ yields $sds = ar\sin\theta d\theta$ and

$$V(r;a) = \frac{1}{2ar}\int_{|a-r|}^{a+r} ds\, U(s)s = \frac{1}{2ar}(f(a+r) - f(|a-r|))\,. \tag{3}$$

Under $U(r) \to U(r) + C$, (C = const) we have $f(r) \to f(r) + Cr^2/2$ and $V(r) \to V(r) + C$. In the case of Coulomb's law, $F(r) \propto r^{-2}$, we have $U(r) \propto r^{-1}$ and $f(r) \propto r$.

(iii) In equilibrium, the potentials on both spheres is the same:

$$Q_1 V(R_1; R_1) + Q_2 V(R_1; R_2) = Q_1 V(R_2; R_1) + Q_2 V(R_2; R_2) \ , \tag{4}$$

and hence

$$\frac{Q_1}{Q_2} = \frac{V(R_2; R_2) - V(R_1; R_2)}{V(R_1; R_1) - V(R_2; R_1)} \ .$$

Plugging in (3) we obtain (1) on the exercise sheet.

Eq. (4) alternatively follows by minimisation of the energy

$$\frac{1}{2}\big(Q_1^2 V(R_1; R_1) + 2Q_1 Q_2 V(R_1; R_2) + Q_2^2 V(R_2; R_2)\big)$$

subject to the constraint $Q_1 + Q_2 = \text{const.}$

$Q_1 = 0$ implies

$$f(2R_2)R_1 - (f(R_2 + R_1) - f(R_2 - R_1))R_2 = 0 \ .$$

Differentiation with respect to R_1, $(0 \leqslant R_1 \leqslant R_2)$ gives

$$f(2R_2) - (f'(R_2 + R_1) + f'(R_2 - R_1))R_2 = 0$$

and in particular

$$f(2R_2) = 2f'(R_2)R_2 \ . \tag{5}$$

On the other hand, renewed differentiation yields

$$f''(R_2 + R_1) - f''(R_2 - R_1) = 0 \ . \tag{6}$$

Equation (6) states that f'' is constant, i.e. $f(r) = C_1 r^2 + C_2 r + C_3$, where $C_3 = 0$ because of (5). This precisely corresponds to the case of Coulomb's law.

Remark. From quantum field theory it follows that Coulomb's law is based on the exchange of virtual photons. If they had a mass $m > 0$, $U(r) \propto r^{-1}$ would have to be replaced by $U(r) \propto r^{-1} \exp^{-(mc/\hbar)r}$. The negative results of Cavendish imply $m < 10^{-40}$g (today $m < 10^{-47}$g with related experiments).

3. Thomson's theorem

Consider charge distributions ρ with $\operatorname{supp} \rho \subset \bigcup_i \mathcal{L}_i$ and given fixed charges $Q_i := \int_{\mathcal{L}_i} \rho(x)\, \mathrm{d}x$. The electrostatic energy which should be minimised is

$$E[\rho] = \frac{1}{8\pi} \int \frac{\rho(x)\rho(y)}{|x - y|}\, \mathrm{d}x\, \mathrm{d}y \ .$$

Let ρ, $\rho + \delta\rho$ be two such distributions; i.e.

$$\int_{\mathcal{L}_i} \delta\rho(x)\, \mathrm{d}x = \int_{\mathcal{L}_i} \delta\rho(x) \cdot 1\, \mathrm{d}x = 0 \ . \tag{7}$$

Then

$$E[\rho + \delta\rho] - E[\rho] = E[\delta\rho] + \int \delta\rho(x)\, \varphi(x)\, \mathrm{d}x \ , \tag{8}$$

where

$$\varphi(x) = \frac{1}{4\pi} \int \frac{\rho(y)}{|x-y|} \,\mathrm{d}y$$

is the electrostatic potential of ρ. Here $E[\delta\rho] \geqslant 0$, cf. (1.20). If we have $\varphi(x) = V_i$ on $\mathcal{L}_i$, then $\int \delta\rho(x)\,\varphi(x)\,\mathrm{d}x = \sum_{i=1}^{N} V_i \int_{\mathcal{L}_i} \delta\rho(x)\,\mathrm{d}x = 0$ and hence $E[\rho+\delta\rho] \geqslant E[\rho]$. On the other hand, if ρ is a minimiser of $E[\cdot]$, (8) and $(\mathrm{d}/\mathrm{d}\lambda)E[\rho+\lambda\delta\rho]\big|_{\lambda=0} = 0$ imply

$$0 = \sum_i \int_{\mathcal{L}_i} \delta\rho(x)\varphi(x)\,\mathrm{d}x$$

for all valid $\delta\rho$. Since $\delta\rho$ can be chosen independently on each body, each term vanishes separately. Equation (7) now implies

$$\varphi(x) = \mathrm{const}\ 1 \quad \text{on } \mathcal{L}_i$$

in the following way: φ is perpendicular to $\delta\rho$ with respect to the scalar product $(u,v) = \int_{\mathcal{L}_i} u(x)v(x)\,\mathrm{d}x$, and the latter are themselves perpendicular to 1 w.r.t. the same scalar product.

4. Electrostatic energy in an external field

The electrostatic energy W of the charge distribution $\rho(x)$ in the external potential $\varphi(x)$ is

$$W = \int \rho(x)\varphi(x)\mathrm{d}x\,. \tag{9}$$

For a nearly constant φ we use the Taylor expansion

$$\begin{aligned}\varphi(x) &= \varphi(0) + \sum_{i=1}^{3} x_i\partial_i\varphi(0) + \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j \partial_i\partial_j\varphi(0) + \ldots \\ &= \varphi(0) - \sum_{i=1}^{3} x_i E_i(0) - \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j \partial_i E_j(0) + \ldots\,.\end{aligned}$$

Then (9) becomes

$$W = e\varphi(0) - p\cdot E(0) - \frac{1}{6}\sum_{i,j=1}^{3} T_{ij}\frac{\partial E_j}{\partial x_i}(0) + \ldots\,,$$

where $T_{ij} = 3\int x_i x_j \rho(x)\,\mathrm{d}x\,.$ Since the external field satisfies $\sum_{i,j=1}^{3}\delta_{ij}(\partial E_j/\partial x_i)(0) = \nabla\cdot E(0) = 0$, T_{ij} can be replaced by $T_{ij} - (\mathrm{Tr}\,T)\delta_{ij}/3 = Q_{ij}$.