Solution 3

1. Helmholtz coil

(i) Let the origin be located at the center of the circle. The magnetic field is given by the Biot-Savart law:

$$B(x) = \frac{I}{4\pi c} \int_\gamma \frac{s \times (x-y)}{|x-y|^3}\, \mathrm{d}y\,.$$

Hence

$$B(-x) = -\frac{I}{4\pi c} \int_\gamma \frac{s \times (x+y)}{|x+y|^3}\, \mathrm{d}y = \frac{I}{4\pi c} \int_\gamma \frac{s \times (x-y)}{|x-y|^3}\, \mathrm{d}y = B(x)$$

by substituting $y \to -y$, where $\gamma \to \gamma$. Similarly, a rotation R around the axis yields

$$B(\mathrm{R}x) = \mathrm{R}B(x)$$

by $y \to \mathrm{R}y$ and $\mathrm{R}a \times \mathrm{R}b = \mathrm{R}(a \times b)$. For $x = (0, 0, x_3)$ we therefore have $B = (0, 0, B_3(x_3))$ with

$$B_3(x_3) = e_3 \cdot B = \frac{I}{4\pi c} 2\pi R \frac{1}{R^2 + x_3^2} \frac{R}{\sqrt{R^2 + x_3^2}} = \frac{I}{2c} \frac{R^2}{(R^2 + x_3^2)^{3/2}}\,. \tag{7}$$

Eq. (7) follows from s and $x - y$ being perpendicular to one another, and the cosine of the angle between its vector product and e_3 being $R/\sqrt{R^2 + x_3^2}$.

(ii) Let B_0 be the field in part (i). The field here is

$$B(x) = B_0(x - \frac{a}{2}e_3) + B_0(x + \frac{a}{2}e_3)\,, \tag{8}$$

where the origin is now located at the center of the two coils. By (i), we have the symmetries

$$B(-x) = B(x)\,, \tag{9}$$
$$B(\mathrm{R}x) = \mathrm{R}B(x)\,. \tag{10}$$

Eq. (9) implies that the Taylor expansion contains only terms of even order. Hence it is enough to check the terms of order two. Eq. (10) implies that B_3 is depending on x_1, x_2 only through $\sqrt{x_1^2 + x_2^2}$. On the symmetry axis we thus have

$$\frac{\partial^2 B_3}{\partial x_1 \partial x_2} = 0\,,$$
$$\frac{\partial^2 B_3}{\partial x_1^2} = \frac{\partial^2 B_3}{\partial x_2^2}\,. \tag{11}$$

By $\partial B_3/\partial x_i = \partial B_i/\partial x_3$, $(i = 1, 2)$ we have there also

$$\frac{\partial^2 B_1}{\partial x_2 \partial x_3} = \frac{\partial^2 B_2}{\partial x_1 \partial x_3} = 0\,,$$
$$\frac{\partial^2 B_1}{\partial x_1 \partial x_3} = \frac{\partial^2 B_2}{\partial x_2 \partial x_3}\,. \tag{12}$$

On the plane in the middle, $x = (x_1, x_2, 0)$, (9, 10) imply for a rotation by π, i.e. $\mathrm{R}x = -x$, $B(x) = \mathrm{R}B(x)$, which implies $B_1 = B_2 = 0$. Hence we have there

$$\left(\frac{\partial^2 B_k}{\partial x_i \partial x_j}\right)^2_{i,j=1} = 0\,, \qquad (k = 1, 2).$$

By the field equation already used and $\nabla \cdot B = 0$ we have

$$\frac{\partial^2 B_1}{\partial x_3^2} = \frac{\partial^2 B_3}{\partial x_1 \partial x_3} = -\frac{\partial}{\partial x_1}\left(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2}\right) = 0\,,$$

and similarly for $1 \leftrightarrow 2$. Only (11, 12) and $\partial^2 B_3/\partial x_3^2$ are not necessarily vanishing at the origin. But we have

$$\frac{\partial^2 B_3}{\partial x_1^2} + \frac{\partial^2 B_3}{\partial x_2^2} = \frac{\partial^2 B_1}{\partial x_1 \partial x_3} + \frac{\partial^2 B_2}{\partial x_2 \partial x_3} = -\frac{\partial^2 B_3}{\partial x_3^2}\,,$$

and hence we only need (1) in order for them to vanish.

(iii) By (7, 8), we have to solve

$$f''(x_3 = 0) = 0$$

w.r.t. a, with

$$f(x_3) = f_0(x_3 - \frac{a}{2}) + f_0(x_3 + \frac{a}{2})\,, \qquad f_0(x_3) = (R^2 + x_3^2)^{-3/2}\,.$$

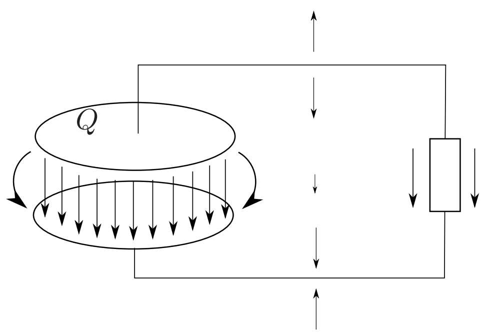
Since f_0 is even, we have to solve $f_0''(a/2) = 0$. By

$$f_0'(x_3) = -3\,\frac{x_3}{(R^2 + x_3^2)^{5/2}}\,,$$
$$f_0''(x_3) = -3\,\frac{1}{(R^2 + x_3^2)^{5/2}} + 15\,\frac{x_3^2}{(R^2 + x_3^2)^{7/2}} = 3\,\frac{4x_3^2 - R^2}{(R^2 + x_3^2)^{7/2}}$$

it follows $a = R$.

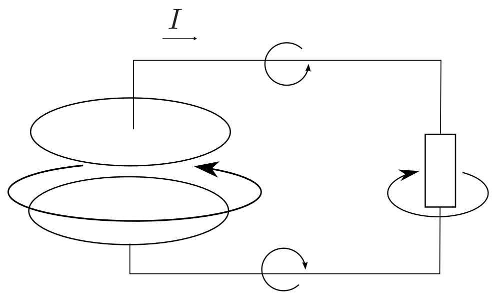
2. Energy flow due to the discharge of a capacitor

If the discharge is slow, the E-field is in every moment approximately given by the one which corresponds to the charges left on the plates being constant.

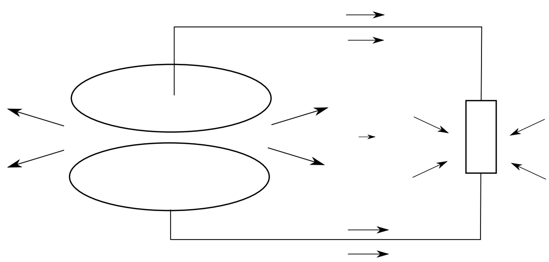


The B-field is generated by the current and the displacement current:

$$\nabla \times B = \frac{1}{c}\Big(\imath + \frac{\partial E}{\partial t}\Big) .$$



The Poynting vector $S = c(E \times B)$ therefore is:



Hence the energy flow from the capacitor to the resistor is located in space and not inside the conductors, although increasingly near them. The conductors guide the field.

3. Completely and partially polarized light

(i) The 2×2 matrices $S = S^*$ are of the general form

$$S = \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix} ,$$

with $a, b, c, d \in \mathbb{R}$. The real vector space V built by them ($(\lambda S)^* = \lambda S$ for $\lambda \in \mathbb{R}!$), therefore has dimension 4. The three Pauli matrices, together with $\sigma_0 = 1_2$, are linearly independent in V and hence $\{\sigma_i\}_{i=0}^3$ is a basis; whence (5).

(ii) This basis is orthonormal w.r.t. the scalar product mentioned in the hint (check: it actually is a scalar product), since

$$(\sigma_i, \sigma_j) = \frac{1}{2}\,\mathrm{Tr}(\sigma_i \sigma_j) = \delta_{ij}\,, \quad (i, j = 0, \dots, 3)$$

by $\mathrm{Tr}\,\sigma_0 = 2$, $\mathrm{Tr}\,\sigma_i = 0$, $(i = 1, 2, 3)$, cf. also the second part of the hint. Thus we have

$$s_i = (\sigma_i, S) = \frac{1}{2}\,\mathrm{Tr}(\sigma_i S)\,, \quad (i = 0, \dots, 3) \tag{13}$$

in (5), and in detail

$$\begin{aligned}
s_0 &= \frac{1}{2}\langle |E_1|^2 + |E_2|^2 \rangle = \frac{1}{2}\langle |\underline{E}|^2 \rangle\,,\\
s_1 &= \frac{1}{2}\langle E_2\overline{E}_1 + E_1\overline{E}_2 \rangle = \langle \operatorname{Re} E_2\overline{E}_1 \rangle\,,\\
s_2 &= -\frac{i}{2}\langle E_2\overline{E}_1 - E_1\overline{E}_2 \rangle = \langle \operatorname{Im} E_2\overline{E}_1 \rangle\,,\\
s_3 &= \frac{1}{2}\langle |E_1|^2 - |E_2|^2 \rangle\,.
\end{aligned} \tag{14}$$

(iii) By $S = \underline{E}\,\underline{E}^*$ we have

$$S^2 = \underline{E}(\underline{E}^*\underline{E})\underline{E}^* = |E|^2 S = 2s_0 S\,.$$

By

$$\frac{1}{2}\operatorname{Tr} S^2 = (S,S) = \sum_{i=0}^{3} s_i^2 = s_0^2 + s^2\,, \quad \frac{1}{2}\operatorname{Tr} S = s_0$$

it follows $s_0^2 + s^2 = 2s_0^2$, i.e. (6).

(iv) By (4, 13) we have

$$s = \frac{1}{2}\operatorname{Tr}(\sigma\langle \underline{E}\,\underline{E}^*\rangle) = \frac{1}{2}\langle \operatorname{Tr}(\sigma\underline{E}\,\underline{E}^*)\rangle\,.$$

Since $|\langle v\rangle| \leqslant \langle |v|\rangle$, it follows

$$|s| \leqslant \frac{1}{2}\langle |\operatorname{Tr}(\sigma\underline{E}\,\underline{E}^*)|\rangle = \frac{1}{2}\langle \operatorname{Tr}\underline{E}\,\underline{E}^*\rangle = s_0\,,$$

where the first equality is (6). Alternatively: for every $\underline{u} \in \mathbb{C}^2$

$$\underline{u}^* S\underline{u} = \underline{u}^*\langle \underline{E}\,\underline{E}^*\rangle\underline{u} = \langle (\underline{u}^*\cdot\underline{E})\overline{(\underline{u}^*\cdot\underline{E})}\rangle \geqslant 0\,,$$

i.e. $S \geqslant 0$ as a matrix. The eigenvalues of

$$S = \begin{pmatrix} s_0 + s_3 & s_1 + is_2 \\ s_1 - is_2 & s_0 - s_3 \end{pmatrix}$$

are given by

$$0 = \det(S - \lambda 1_2) = \lambda^2 - 2s_0\lambda + (s_0^2 - s^2) = (\lambda - (s_0 + |s|))(\lambda - (s_0 - |s|))\,,$$

i.e. $\lambda_\pm = s_0 \pm |s|$. They are $\geqslant 0$ since $S \geqslant 0$.

(v) The eigenvalues of σ_i are ± 1, since $\sigma^2 = 1, \operatorname{Tr}\sigma_i = 0$. The normed eigenvectors, $\sigma_i e_\pm^{(i)} = \pm e_\pm^{(i)}$, are

$$\begin{aligned}
e_\pm^{(1)} &= \frac{1}{\sqrt{2}}(e_1 \pm e_2) && (\pm 45^\circ\text{- polarization})\\
e_\pm^{(2)} &= \frac{1}{\sqrt{2}}(e_1 \pm ie_2) && (\text{right-, left-circular})\\
e_+^{(3)} &= e_1\,, \quad e_-^{(3)} = e_2 && (\text{horizontal, vertical})\,.
\end{aligned} \tag{15}$$

The matrix σ_i is described by $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ w.r.t. the eigenbasis $\{e_+^{(i)}, e_-^{(i)}\}$, and S by $\langle\alpha_k\overline{\alpha}_j\rangle_{k,j=\pm}$. Since the trace is independent of the basis, it follows, as in (14),

$$s_i = \frac{1}{2}\langle|\alpha_+^{(i)}|^2 - |\alpha_-^{(i)}|^2\rangle\,,$$
$$s_0 = \frac{1}{2}\langle|\alpha_+^{(i)}|^2 + |\alpha_-^{(i)}|^2\rangle\,.$$

Thus $c \cdot s_0$ is the intensity, and $-1 \leqslant s_i/s_0 \leqslant 1$, $(i = 1, 2, 3)$ describes the relative share of $(+/-)$-polarizations in the intensity w.r.t. the three bases (15). A wave with $s = 0$ is unpolarized w.r.t. all of them (example: direct sunlight, in a good approximation).