Solution 4

1. Solution of the inhomogeneous wave equation

(i) Let $\hat{f}$ denote the Fourier transform of f. Then the Fourier transform of the inhomogeneous wave equation reads

$$\frac{1}{c^2}\partial_t^2 \hat{u}(t,k) = -k^2 \hat{u}(t,k) + \hat{f}(t,k).$$

Define $\hat{v}(t,k) = \partial_t \hat{u}(t,k)$. Then

$$\partial_t \hat{v}(t,k) = c^2(-k^2\hat{u}(t,k) + \hat{f}(t,k)),$$

and hence

$$\partial_t \begin{pmatrix} \hat{u}(t,k) \\ \hat{v}(t,k) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{pmatrix}}_{=:A(k)} \begin{pmatrix} \hat{u}(t,k) \\ \hat{v}(t,k) \end{pmatrix} + \begin{pmatrix} 0 \\ c^2\hat{f}(t,k) \end{pmatrix}. \tag{2}$$

By Duhamel's principle, the solution of (2) is

$$\begin{pmatrix} \hat{u}(t,k) \\ \hat{v}(t,k) \end{pmatrix} = \exp(tA(k)) \begin{pmatrix} \hat{u}(0,k) \\ \hat{v}(0,k) \end{pmatrix} + \int_0^t \mathrm{d}s\, \exp((t-s)A(k)) \begin{pmatrix} 0 \\ c^2\hat{f}(s,k) \end{pmatrix}, \tag{3}$$

with

$$\exp(tA(k)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A(k)^n.$$

Noting that

$$A(k)^{2n} = (-1)^n (c|k|)^{2n}\mathbb{1}, \quad A(k)^{2n+1} = (-1)^n (c|k|)^{2n+1} \begin{pmatrix} 0 & 1/(c|k|) \\ -c|k| & 0 \end{pmatrix},$$

we can write

$$\begin{aligned} \exp(tA(k)) &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} (c|k|)^{2n}\mathbb{1} + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} (c|k|)^{2n+1} \begin{pmatrix} 0 & 1/(c|k|) \\ -c|k| & 0 \end{pmatrix} \\ &= \cos(c|k|t)\mathbb{1} + \sin(c|k|t) \begin{pmatrix} 0 & 1/(c|k|) \\ -c|k| & 0 \end{pmatrix}. \end{aligned}$$

Thus we have

$$\hat{u}(t,k) = \cos(c|k|t)\hat{u}(0,k) + \frac{\sin(c|k|t)}{c|k|}\hat{v}(0,k) + \int_0^t \mathrm{d}s\, \frac{\sin(c|k|(t-s))}{c|k|} c^2 \hat{f}(s,k). \tag{4}$$

Note that $\hat{v}(0,k)$ is the Fourier transform of $\partial_t u(0,x)$. Furthermore, for $t > 0$

$$\frac{\sin(c|k|t)}{c|k|} = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(0)} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}k\cdot y} = \int_{\mathbb{R}^3} \mathrm{d}y\, \frac{\delta(|y|-ct)}{4\pi c^2 t} \mathrm{e}^{-\mathrm{i}k\cdot y} = \int_{\mathbb{R}^3} \mathrm{d}y\, \frac{\delta(|y|-ct)}{4\pi c|y|} \mathrm{e}^{-\mathrm{i}k\cdot y}$$

by (1), and hence $\sin(c|k|t)/(c|k|)$ is the Fourier transform of $\delta(|y|-ct)/(4\pi c|y|)$. For $t<0$ we have $\sin(c|k|t) = -\sin(c|k|(-t))$ which leads to $-\delta(|y|+ct)/(4\pi c|y|)$. Moreover, we have $\cos(c|k|t) = \partial_t \sin(c|k|t)/(c|k|)$, which implies that it is the Fourier transform of $\partial_t\delta(|y|-ct)/(4\pi c|y|)$ and $-\partial_t\delta(|y|+ct)/(4\pi c|y|)$ respectively. We therefore get

$$
\begin{aligned}
u(t,x) = \int_{\mathbb{R}^3} \mathrm{d}y \Bigg[& \frac{1}{c}\partial_t \left(\frac{\delta(|y|-ct)}{4\pi|y|} - \frac{\delta(|y|+ct)}{4\pi|y|} \right) u(0,x-y) \\
& + \left(\frac{\delta(|y|-ct)}{4\pi|y|} - \frac{\delta(|y|+ct)}{4\pi|y|} \right) \frac{1}{c}\partial_t u(0,x-y) \\
& + \int_0^t \mathrm{d}s \left(\frac{\delta(|y|-c(t-s))}{4\pi|y|} - \frac{\delta(|y|+c(t-s))}{4\pi|y|} \right) c\, f(s,x-y) \Bigg].
\end{aligned} \tag{5}
$$

Eq. (1) is verified as follows:

$$
\begin{aligned}
\frac{1}{4\pi r^2} \int_{\partial B_r(0)} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}k\cdot y} &= \frac{1}{4\pi r^2} \int_{\mathbb{R}^3} \mathrm{d}y\, \delta(|y|-r) \mathrm{e}^{-\mathrm{i}k\cdot y} \\
&= \frac{1}{2r^2} \int_0^\pi \mathrm{d}\theta\, \sin(\theta) \int_0^\infty \mathrm{d}\tilde{r}\, \tilde{r}^2\, \delta(\tilde{r}-r) \mathrm{e}^{-\mathrm{i}|k|\tilde{r}\cos(\theta)} \\
&= \frac{1}{2} \int_0^\pi \mathrm{d}\theta\, \underbrace{\sin(\theta) \mathrm{e}^{-\mathrm{i}|k|r\cos(\theta)}}_{=\frac{1}{\mathrm{i}r|k|}\partial_\theta \mathrm{e}^{-\mathrm{i}|k|r\cos(\theta)}} \\
&= \frac{1}{r|k|} \frac{1}{2i} (\mathrm{e}^{\mathrm{i}|k|r} - \mathrm{e}^{-\mathrm{i}|k|r}) .
\end{aligned}
$$

(ii) For $u(0,\cdot) = \partial_t u(0,\cdot) = 0$ and $f(s,x) = 0$ for $s \leqslant 0$, (5) becomes

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^3} \mathrm{d}y \int_0^t \mathrm{d}s\, \frac{\delta(|y|-c(t-s))}{4\pi|y|}\, c\, f(s,x-y) \\
&= \int_{\mathbb{R}^3} \mathrm{d}y \int_{-\infty}^t \mathrm{d}s\, \frac{\delta(\frac{|y|}{c}-(t-s))}{4\pi|y|} f(s,x-y) \\
&= \int_{\mathbb{R}^3} \mathrm{d}y \int_{-\infty}^\infty \mathrm{d}s\, \Theta(r-s)\, \frac{\delta(\frac{|y|}{c}-(t-s))}{4\pi|y|} f(s,x-y) \\
&= \int_{\mathbb{R}^3} \mathrm{d}y\, \frac{f(t-\frac{|y|}{c}, x-y)}{4\pi|y|} ,
\end{aligned}
$$

where Θ denotes the Heaviside function. In the second step, the integral can be extended to $-\infty$ since $f(s,x) = 0$ for $s \leqslant 0$, and the delta function is rescaled.

2. Solution of the wave equation in two dimensions

(i) The solution of the 3-dimensional wave equation $\Box u(t,x) = 0$ with initial conditions independent of x_3,

$$
u(0,x) = u(0,\underline{x}) , \qquad \partial_t u(0,x) = \partial_t u(0,\underline{x}) , \qquad (x = (\underline{x}, x_3)) ,
$$

is itself independent of x_3, $u(t,x) = u(t,\underline{x})$, and solves the initial value problem in dimension 2. The distributional solution $D_2(t,\underline{x})$ is the solution with initial conditions

$$D_2(0,\underline{x}) = 0, \qquad \frac{1}{c}\partial_t D_2(0,\underline{x}) = \delta^{(2)}(\underline{x})\,.$$

It is therefore found via the 3-dimensional distributional solution $D_3(t,x)$:

$$D_2(t,\underline{x}) = \int \mathrm{d}y\, D_3(t,x-y)\delta^{(2)}(\underline{y}) = \int \mathrm{d}y_3\, D_3(t,\underline{x},-y_3)\,.$$

Here x_3 is set to 0, which is valid since we noticed before that the expression is independent of x_3. For $t \geq 0$ we have

$$D_2(t,\underline{x}) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}y_3\, \frac{\delta\big(ct - \sqrt{\underline{x}^2 + y_3^2}\big)}{\sqrt{\underline{x}^2 + y_3^2}} = \frac{1}{2\pi}\int_{|\underline{x}|}^{\infty} \mathrm{d}r\, \frac{\delta(ct-r)}{\sqrt{r^2 - \underline{x}^2}} = \frac{1}{2\pi}\frac{\theta(ct - |\underline{x}|)}{\sqrt{c^2t^2 - \underline{x}^2}}$$

with the substitution $r^2 = \underline{x}^2 + y_3^2$ (for $y_3 > 0$ and $y_3 < 0$), under which

$$\frac{\mathrm{d}y_3}{r} = \frac{\mathrm{d}r}{y_3} = \frac{\mathrm{d}r}{\sqrt{r^2 - \underline{x}^2}}\,.$$

Together with a similar result for $t \leq 0$ this implies

$$D_2(t,\underline{x}) = \frac{1}{2\pi\sqrt{c^2t^2 - |\underline{x}^2|}}\big(\theta(ct - |\underline{x}|) - \theta(-ct - |\underline{x}|)\big)$$

(ii) The support of $D_2(t,\underline{x})$ is $|\underline{x}| \leqslant c|t|$ (instead of $|x| = ct$ for $D_3(t,x)$). Thus $u(t,\underline{x})$ depends on $u(\tilde{t},\tilde{\underline{x}})$ for $|\underline{x} - \tilde{\underline{x}}| \leqslant c|t - \tilde{t}|$. Interpretation: a wave which is localized at $\underline{x} = 0$ at time $t = 0$ is noticed for the first time at position $\underline{x}$ at time $|\underline{x}|/c$, but afterwards fades away only slowly.