Solution 5

1. Hertzian dipole

(i)

$$\frac{\partial\rho}{\partial t} = -\dot{p}(t)\cdot\nabla\delta(x) = -\nabla\cdot\imath\,.$$

(ii) We have $\int \nabla\delta(x)\cdot f(x)\,\mathrm{d}x = -(\nabla\cdot f)(0)$ for a vector-valued function f, and hence

$$\begin{aligned}\varphi(x,t) &= \frac{1}{4\pi}\int \mathrm{d}y\,\frac{\rho\big(y,t-\frac{|x-y|}{c}\big)}{|x-y|} = \frac{1}{4\pi}\,\nabla_y\cdot\Big(\frac{1}{|x-y|}p\big(t-\frac{|x-y|}{c}\big)\Big)\Big|_{y=0}\\ &= -\frac{1}{4\pi}\nabla\cdot\Big(\frac{1}{r}p\big(t-\frac{r}{c}\big)\Big) = \frac{1}{4\pi}\Big(\frac{x\cdot\dot{p}}{cr^2}+\frac{x\cdot p}{r^3}\Big)\,,\qquad (r=|x|)\,.\end{aligned}$$

We used the product and chain rule in the last step, where p and $\dot{p}$ have to be taken at the retarded time $t-r/c$. Furthermore we have

$$A(x,t) = \frac{1}{4\pi c}\int \mathrm{d}y\,\frac{\imath\big(y,t-\frac{|x-y|}{c}\big)}{|x-y|} = \frac{1}{4\pi c r}\dot{p}(t-\frac{r}{c})\,.$$

The same rules then imply

$$\begin{aligned}4\pi B &= e\wedge\Big(-\frac{1}{c^2 r}\ddot{p}-\frac{1}{cr^2}\dot{p}\Big)\,,\\ 4\pi E &= \frac{1}{c^2 r}\big((\ddot{p}\cdot e)e-\ddot{p}\big)+\frac{1}{cr^2}\big(3(\dot{p}\cdot e)e-\dot{p}\big)+\frac{1}{r^3}\big(3(p\cdot e)e-p\big)\,,\end{aligned}$$

where $e = x/r$.

(iii) In this case B is perpendicular to e and p, whereas E is a linear combination of e and p and therefore lies in the plane spanned by them.

(iv) In the time-harmonic case we have $\rho,\imath\sim e^{i\omega t}$, $\lambda = 2\pi c/\omega$, i.e. $p(t) = p_0 e^{i\omega t}$ for some time-independent vector p_0. Thus the terms of B and E are of the order of magnitude

$$\frac{1}{r\lambda^2},\ \frac{1}{r^2\lambda}\quad\text{and}\quad\frac{1}{r\lambda^2},\ \frac{1}{r^2\lambda},\ \frac{1}{r^3}$$

respectively. For $r\ll\lambda$ the last terms dominate (for E this is the (retarded) electrostatic field); for $r\gg\lambda$ the first terms dominate. In the first case we have $B\propto\dot{p}$ and $E\propto p$, i.e. the fields have a relative phase of 90° ($-i = e^{-i\frac{\pi}{2}}$); in the second case ($B,E\propto\ddot{p}$) they have the same phase.

(v) The above A-field is the one of the electric dipole radiation; and in particular this is not an approximation, since the extent of the source is $d=0$. The same holds for the Poynting vector and the power.

2. Thomson scattering

The motion of the particle is due to the Lorentz force exerted by the electric and magnetic components of the incident wave. Since the latter is periodic in time, so is the motion of the particle. The equation of motion for the particle is

$$m\ddot{x} = e(E + \frac{\dot{x}}{c} \times B)\,. \tag{1}$$

Neglecting the field of the particle itself we have

$$E(x,t) = \mathrm{Re}(E_0 e^{i(k\cdot x - \omega t)}), \quad B = \hat{e} \times E\,,$$

where $k = (\omega/c)\hat{e}$ is the wave vector. The intensity

$$I_0 = \frac{c}{2}(E_0, E_0)$$

is assumed to be small, which implies that the amplitude of the wave $|E_0|$ is small.

The particle is assumed to be initially at rest at $x = 0$ and then to start small oscillations around $x = 0$ due to the incident wave. Hence we have $|x|$ and $|\dot{x}|/c$ of the order of $|E_0|$. Thus the part of the magnetic field in (1) is of order two in $|E_0|$ and therefore neglected. Moreover, if $|x| \ll \lambda$ we have $k \cdot x = (\omega|x|/c)\cos\sphericalangle(k,x) = (2\pi|x|/\lambda)\cos\sphericalangle(k,x) \ll 1$, and we can set $x = 0$ in the exponent. Thus in this case, (1) to first order in $|E_0|$ simplifies to

$$m\ddot{x} = e\mathrm{Re}(E_0 e^{-i\omega t})\,,$$

which admits the solution

$$x = -\frac{e}{m\omega^2}\mathrm{Re}(E_0 e^{-i\omega t})\,.$$

The assumption therefore requires $e|E_0| \ll m\omega^2\lambda = 2\pi m\omega c$.

For $|x| \ll \lambda$ the electric dipole radiation is dominant, with $p(t) = \int y\, e\, \delta(y - x)\, \mathrm{d}y = ex$ and hence $\ddot{p} = e\ddot{x} = (e^2/m)\mathrm{Re}(E_0 e^{-i\omega t})$. The radiated power averaged over time is thus

$$P = \frac{1}{6\pi c^3}\frac{e^4}{m^2}\frac{1}{2}(E_0, E_0)\,,$$

and we get

$$\sigma = \frac{e^4}{6\pi c^4 m^2}\,.$$

Remark: The particle absorbs energy from the incident wave and re-emits it as electromagnetic radiation. Such a process is equivalent to the scattering of the electromagnetic wave by the particle. More precisely, intensity of the scattered radiation is such as if the incident wave, flowing through a section of area σ, would be scattered.

3. The Michelson-Morley experiment

(i) Elapsed time SS_2S: In the laboratory system, the velocities for the way to S_2 and back are $c - v$ and $c + v$ respectively, hence

$$t_2 = d_2(\frac{1}{c-v} + \frac{1}{c+v}) = \frac{2d_2}{c}\frac{1}{1 - v^2/c^2}\,.$$

Elapsed time SS_1S: In the reference system of the aether, S is shifted by vt_1 and the light has travelled the distance $2\sqrt{d_1^2 + (vt_1/2)^2}$ (Pythagoras). Hence $4d_1^2 + v_1^2 t_1^2 = c^2 t_1^2$, i.e.

$$t_1 = \frac{2d_1}{c}\frac{1}{\sqrt{1-v^2/c^2}}\,.$$

Consequently

$$\Delta t = t_2 - t_1 = \frac{2}{c}\Big(\frac{d_2}{1-v^2/c^2} - \frac{d_1}{\sqrt{1-v^2/c^2}}\Big) = \frac{2}{c}\Big((d_2-d_1) + \frac{v^2}{c^2}(d_2 - \frac{d_1}{2}) + O((\frac{v}{c})^4)\Big)\,.$$

For the rotated apparatus we have

$$t_2' = \frac{2d_2}{c}\frac{1}{\sqrt{1-v^2/c^2}}\,, \qquad t_1' = \frac{2d_1}{c}\frac{1}{1-v^2/c^2}$$

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\Big((d_2-d_1) + \frac{v^2}{c^2}(\frac{d_2}{2} - d_1) + \ldots\Big)\,.$$

We conclude

$$\Delta t' - \Delta t = -\frac{2}{c}\frac{v^2}{c^2}\Big(\frac{d_2}{2} + \frac{d_1}{2}\Big) = -\frac{d_1+d_2}{c}\Big(\frac{v}{c}\Big)^2.$$

One wave is shifted w.r.t. the other by

$$\frac{c|\Delta t' - \Delta t|}{\lambda} = \frac{d_1+d_2}{\lambda}\Big(\frac{v}{c}\Big)^2 \tag{2}$$

in units of the wavelength. The interference pattern is shifted by the same fraction of the distance between the stripes.

(ii) With $c \approx 3 \cdot 10^8$m/s we have $v/c \approx 10^{-4}$, and (2) is $\approx 10^{-1} = 10\%$.