Solution 6

1. Applications of Lorentz transformations

For all subtasks we may (or should) assume that the coordinates of an event w.r.t. O' are given by those w.r.t. O after the application of a boost in 1-direction. The latter is given by

$$\Lambda(v\underline{e}_1) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{1}$$

where $\beta = v/c$ (with $|\beta| < 1$) and $\gamma = (1-\beta^2)^{-1/2}$.

(i) Let $(ct_A, \underline{x})$ and $(ct_B, \underline{x})$ be the coordinates of the events A and B in O (note the common position $\underline{x} = (x^1, x^2, x^3)$). Their time coordinates w.r.t. O' are

$$ct'_A = \gamma ct_A - \beta\gamma x^1 , \qquad ct'_B = \gamma ct_B - \beta\gamma x^1 .$$

Thus the time difference between A and B in O' is

$$\Delta t' = t'_B - t'_A = \gamma(t_B - t_A) = \frac{\Delta t}{\sqrt{1-(v/c)^2}} \geqslant \Delta t .$$

(ii) In the rest frame O of the rod, its end points have the fixed spatial coordinates $\underline{x}_A = (x_A, 0, 0)$ and $\underline{x}_B = (x_B, 0, 0)$ (rod in 1-direction). In the inertial system O' moving longitudinal w.r.t. the rod, denote the coordinates (measured at the same time w.r.t. O'!) of its end points by $(ct', x'_A, 0, 0)$ and $(ct', x'_B, 0, 0)$. The coordinates of these events in O follow by the application of the matrix $\Lambda(-v\underline{e}_1)$. In particular, for the 1-coordinates we have

$$x_A = \beta\gamma ct' + \gamma x'_A , \qquad x_B = \beta\gamma ct' + \gamma x'_B .$$

If L' denotes the length of the rod in the system O', then

$$L = x_B - x_A = \gamma(x'_B - x'_A) = \gamma L' ,$$

and hence $L' = \sqrt{1-(v/c)^2}L \leqslant L$. (Note that it is not relevant that the two events are not at the same time in O, since the rod is at rest w.r.t. this system.)

(iii) The relation between the coordinates $(ct, \underline{x})$, $(ct', \underline{x}')$ of an event w.r.t. the two inertial systems is

$$ct = \gamma(ct' + \beta x'^1) , \qquad x^1 = \gamma(x'^1 + \beta ct') , \qquad x^k = x'^k , \qquad (k = 2, 3) .$$

By definition, the measurement of the position of the end points of the rod w.r.t. O' happens there at the same time: $\Delta t' = 0$. Therefore $c\Delta t = \gamma\beta\Delta x'^1$ and $\Delta x'^2 = \Delta x^2 = w\Delta t$. Hence

$$\tan\theta = \frac{\Delta x'^2}{\Delta x'^1} = \gamma\frac{wv}{c^2} .$$

(iv) After a possible rotation of the spatial coordinates, and a possible translation of the spatial and time coordinates, we may assume that the events x, y have the coordinates

$$x = (0,0,0,0)\,, \qquad y = (y^0, y^1, 0, 0)$$

w.r.t. the system O. Then the squared Minkowski norm of $\xi = x - y$ is

$$\langle x - y, x - y\rangle = (y^0)^2 - (y^1)^2\,. \tag{2}$$

We clearly have $\langle \xi, \xi\rangle > 0$ (resp. < 0) if $y^1 = 0$ (resp. $y^0 = 0$), i.e. if the events take place at the same position (resp. the same time).

The converse is also true: in a system O', which emerges from O through (1), the coordinates of the two events are

$$(x')^\mu = (0,0,0,0)\,, \qquad (y')^\mu = (\gamma(y^0 - \beta y^1)\,, \gamma(y^1 - \beta y^0), 0, 0)\,. \tag{3}$$

If $x - y$ is spacelike, (2) implies $|y^0|/|y^1| < 1$ and we can choose $\beta = y^0/y^1$. By (3), the two events then take place at the same time in the new system.

On the other hand, if $x - y$ is timelike, then $|y^1|/|y^0| < 1$. By choosing $\beta = y^1/y^0$, the two events now take place at the same position.

Remark: It is convenient to deduce relations between inertial systems from the Lorentz transformation (1). However, it is often enough to consider the fundamental invariant $\langle \xi, \xi\rangle = (\xi^0)^2 - \underline{\xi}^2$. E.g. for part (i): in O, the position of the clock follows an inertial trajectory $\underline{x} = \underline{v}t + \underline{b}$, with $\underline{v} \equiv 0$ and hence $\underline{b} \equiv \underline{x}$. By exercise 2, the position of the clock in O' follows an inertial trajectory as well, i.e. $\underline{x}' = \underline{v}'t' + \underline{b}'$, but now $\underline{v}' \neq 0$ in general. Hence if ξ is the difference of the 4-vectors of the two events, we have

$$\langle \xi, \xi\rangle = c^2(\Delta t')^2 - (\Delta \underline{x}')^2 = c^2(\Delta t)^2 - (\Delta \underline{x})^2\,,$$

which implies the statement, since $|\Delta \underline{x}| = 0$ and $|\Delta \underline{x}'| = v'|\Delta t'|$.

Another example is part (ii): the length L' of the rod can as well be determined in terms of the time difference T' between the events defined by the two ends passing a fixed mark: $L' = vT'$. From this perspective we have $\Delta \underline{x}' = 0$, $\Delta t' = T'$. Since the mark has the velocity v w.r.t. O, we have $|\Delta \underline{x}| = v|\Delta t|$; and moreover $|\Delta \underline{x}| = L$. Thus by

$$c^2(\Delta t)^2 - (\Delta \underline{x})^2 = c^2(\Delta t')^2 - (\Delta \underline{x}')^2\,,$$

we have $((c^2/v^2) - 1)L^2 = c^2 T'^2$ and $L'^2 = (1 - v^2/c^2)L^2$, as before.

2. Lorentz transformations on the celestial sphere

First note that it is enough to show that the map S is a Möbius transformation for every $\Lambda \in L_+^\uparrow$ in order to deduce the stated isomorphism, since the two groups have the same dimension (namely 6). Furthermore, note that Lorentz transformations in $L_+^\uparrow$ preserve the spatial orientation, and hence S preserves the orientation as well. We are thus left to show that S maps circles on S^2 to circles on S^2.

The condition that the points $\underline{v}_1, \underline{v}_2, \underline{v}_3$ lie on a line is equivalent to the following condition: inertial trajectories which emerge from a common event through these velocities lie on the same plane in the spacetime $\mathbb{R}^4$. Indeed, if $(t_0, \underline{b})$ is the common event, the trajectories

$\underline{x}_i(t) = \underline{b} + \underline{v}_i(t - t_0)$ read $t \mapsto (t, \underline{x}_i(t))$, $(c = 1)$, when conceived in $\mathbb{R}^4$, and have (constant) tangential vectors $\binom{1}{\underline{v}_i}$. We have

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ \underline{v}_1 - \underline{v}_3 & \underline{v}_2 - \underline{v}_3 & \underline{v}_3 \end{pmatrix}$$
$$= \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ \underline{v}_1 - \underline{v}_3 & \underline{v}_2 - \underline{v}_3 & 0 \end{pmatrix} = 1 + \operatorname{rank}\begin{pmatrix} \underline{v}_1 - \underline{v}_3 & \underline{v}_2 - \underline{v}_3 \end{pmatrix} .$$

The two conditions are now seen to be equivalent, since they correspond to the case where one of the ends of the equation is equal to 2.

By (i), we have $\Lambda(1, \underline{v}_i) = (t', \underline{b} + \underline{v}_i' t')$. Moreover, Lorentz transformations are affine and therefore map planes to planes, which implies that the transformed vectors lie on a plane as well. Since $\underline{b}$ and t' are the same for all three vectors, we deduce that the second form of the condition is invariant under Lorentz transformations. Hence S maps lines to lines and therefore also planes to planes.

By (ii), S leaves the ball S^2 invariant. Circles on it are intersections with a plane in $\mathbb{R}^3$, and hence are mapped circles: S is a Möbius transformation.

3. Doppler shift and aberration

(i) Let $x = (ct, \underline{x})$ and $k = (\omega/c, \underline{k})$. Then $\varphi(x) = e^{-i\langle k,x\rangle}$, where $\langle\cdot,\cdot\rangle$ denotes the Minkowski scalar product. The transformed field is thus

$$\varphi'(x') = \varphi(\Lambda^{-1}x') = e^{-i\langle k,\Lambda^{-1}x'\rangle} = e^{-i\langle \Lambda k,x'\rangle} = e^{-i\langle k',x'\rangle} ,$$

with $k' = \Lambda k$, and the third equality following from the invariance of the scalar product under Lorentz transformations. Hence $k = (\omega/c, \underline{k})$ transforms like a 4-vector under Lorentz transformations.

(ii) The 4-wavevector of the light has components $k = (\omega/c, \omega/c, 0, 0)$ w.r.t. O, since $|\underline{k}| = \omega/c$. Its components w.r.t. O' are computed by applying matrix (1),

$$k' = \Lambda(v\underline{e}_1)k = ((\gamma\omega - \beta\gamma\omega)/c, (-\beta\gamma\omega + \gamma\omega)/c, 0, 0) ;$$

and we deduce

$$\omega' = \omega\gamma(1 - \beta) = \omega\sqrt{\frac{1 - v/c}{1 + v/c}} .$$

Remark: To first order in v/c we have $\omega' = \omega(1 - v/c)$, which is the non-relativistic result.

(iii) We apply now (1) to the 4-vector $k = (\omega/c, 0, \omega/c, 0)$ and find

$$k' = (\gamma\omega/c, -\beta\gamma\omega/c, \omega/c, 0)$$

w.r.t. O'. Thus

$$\tan\alpha(v) = k'^1/k'^2 = -\beta\gamma = -\frac{v/c}{\sqrt{1 - (v/c)^2}} .$$

4. Transformation of velocities

The 4-velocity of the particle is $u = \gamma(w)(c, \underline{w})$, where $\gamma(w) = 1/\sqrt{1 - w^2/c^2}$. Since u is a 4-vector, its components transform as

$$u'^{\,0} = \gamma(u^0 - \beta u^1)\,, \quad u'^{\,1} = \gamma(u^1 - \beta u^0)\,, \quad u'^{\,k} = u^k\,, \quad (k = 2, 3)$$

under the boost (1). Hence

$$u' = \gamma(w')(c, \underline{w'}) = (\gamma(u^0 - \beta u^1), \gamma(u^1 - \beta u^0), u^2, u^3)\,,$$

and thus

$$w'_1 = u'^{\,1}\frac{c}{u'^{\,0}} = c\frac{u^1 - \beta u^0}{u^0 - \beta u^1} = \frac{w_1 - v}{1 - vw_1/c^2}\,, \quad w'_k = u'^{\,k}\frac{c}{u'^{\,0}} = c\frac{u^k}{\gamma(u^0 - \beta u^1)} = \frac{w_k}{\gamma(1 - vw_1/c^2)}\,,$$

$(k = 2, 3)$.