## Solution 7

## 1. Field of a uniformly moving charge

Let x' be the coordinates w.r.t. the rest frame of the particle, i.e.  $x' = \Lambda x$  with  $\Lambda = \Lambda(v)$  a boost in 1-direction. In particular

$$|\underline{x}'| = \gamma \left( (x^1 - vt)^2 + \left(1 - \frac{v^2}{c^2}\right) \left( (x^2)^2 + (x^3)^2 \right) \right)^{1/2} =: \gamma \rho(x)$$

We have

$$\underline{E}'(\underline{x}') = \frac{e}{4\pi |\underline{x}'|^3} \underline{x}' \,, \qquad \underline{B}'(\underline{x}') = 0 \,.$$

To transform back to the original frame we have to apply the boost  $\Lambda^{-1} = \Lambda(-v)$ . From the transformation law for the electric field under a boost we get

$$E_1(x) = E'_1(x') = \frac{e}{4\pi\gamma^3\rho^3}\gamma(x^1 - vt) = \frac{e}{4\pi\rho^3}\left(1 - \frac{v^2}{c^2}\right)(x^1 - vt),$$
  

$$E_i(x) = \gamma E'_i(x') = \frac{e}{4\pi\rho^3}\left(1 - \frac{v^2}{c^2}\right)x^i, \qquad (i = 2, 3).$$

For  $\underline{B}$  one can proceed in the same way. An alternative way is to consider the reverse transformation, which immediately yields the solution

$$\underline{B} = \frac{\underline{v}}{c} \wedge \underline{E} \,,$$

since  $\underline{B}' = 0$ . Note: the  $\underline{E}$  field is pointing away from the instantaneous position of the particle, and not from the retarded one. We have

$$|\underline{E}(\underline{x}, t=0)| = \frac{e}{\rho^3} \left(1 - \frac{v^2}{c^2}\right) |\underline{x}|.$$

For  $|\underline{x}|$  fixed, the expression

$$\rho(\underline{x}, t = 0) = (x^1)^2 + \left(1 - \frac{v^2}{c^2}\right)\left((x^2)^2 + (x^3)^2\right)$$

is minimal for  $x^1 = 0$  ( $\underline{x} \perp \underline{v}$ ); and maximal for  $x^2 = x^3 = 0$  ( $\underline{x} \parallel \underline{v}$ ). Hence the reverse holds true for  $|\underline{E}|$ .

## 2. Dual field tensor

(i) Let  $\mu\nu\rho\sigma$  be a fixed permutation of 0123, with  $\varepsilon_{\mu\nu\rho\sigma} = +1$ . For fixed  $\rho$ ,  $\sigma$  in (1), only one additional permutation contributes to the sum, namely  $\nu\mu\rho\sigma$ . Thus (without summation over  $\mu\nu$ )

$$\mathcal{F}_{\rho\sigma} = \frac{1}{2} \left( F^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} + F^{\nu\mu} \varepsilon_{\nu\mu\rho\sigma} \right) = F^{\mu\nu} \,. \tag{3}$$

In particular,

$$\mathcal{F}_{0i} = F^{i+1\,i+2} = -B_i, \qquad \mathcal{F}_{i+1\,i+2} = F^{0i} = -E_i, \qquad (\text{mod }3),$$

since  $(0123) \mapsto (0ii + 1i + 2)$  is even. Hence

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix},$$

which is alternatively achieved by the substitution  $(\underline{E}, \underline{B}) \rightsquigarrow (-\underline{B}, \underline{E})$  in  $F_{\mu\nu}$ .

(ii) It is enough to show the hint for  $\mu\nu\rho\sigma$  as above. With  $\varepsilon_{\mu\nu\rho\sigma} = \varepsilon_{\sigma\nu\mu\rho} = \varepsilon_{\rho\nu\sigma\mu} = +1$ , we can accomplish (3) to

$$\mathcal{F}_{\rho\sigma} = F^{\mu\nu}, \qquad \mathcal{F}_{\mu\rho} = F^{\sigma\nu}, \qquad \mathcal{F}_{\sigma\mu} = F^{\rho\nu}, \qquad (4)$$

i.e. (without summation in the expression in the middle)

$$\mathcal{F}_{\rho\sigma,\mu} + \mathcal{F}_{\mu\rho,\sigma} + \mathcal{F}_{\sigma\mu,\rho} = F^{\mu\nu}{}_{,\mu} + F^{\sigma\nu}{}_{,\sigma} + F^{\rho\nu}{}_{,\rho} = F^{\alpha\nu}{}_{,\alpha},$$

which confirms equation (2). By (1), we have

$$\mathcal{F}^{\rho\sigma} = \frac{1}{2} F_{\mu\nu} \varepsilon^{\mu\nu\rho\sigma}$$

where  $\varepsilon^{0123} = 1 \cdot (-1)^3 \varepsilon_{0123} = -1$  by raising all indices. Hence (4) holds with opposite indices up to the sign. Thus

$$F_{\rho\sigma,\mu} + F_{\mu\rho,\sigma} + F_{\sigma\mu,\rho} = -\mathcal{F}^{\alpha\nu}{}_{,\alpha}\varepsilon_{\mu\nu\rho\sigma} \,. \tag{5}$$

From (2, 5) and Maxwell's equations for F, those for  $\mathcal{F}$  can be deduced.

## 3. Seeing is not measuring

For inertial trajectories, to be parallel in spacetime  $\mathbb{R}^4$  is equivalent to being parallel in space and additionally having the same velocity; e.g. c for light. This follows from:  $(ct, \underline{b} + \underline{v}t)$  and  $(c\tilde{t}, \tilde{\underline{b}} + \tilde{v}\tilde{t})$  are parallel in  $\mathbb{R}^4 \Leftrightarrow (ct, \underline{b} + \underline{v}t) = (ck\tilde{t}, \tilde{\underline{b}} + k\tilde{\underline{v}}\tilde{t})$  for some  $k \in \mathbb{R}$ ; and the equality for the first coordinate implies  $k = t/\tilde{t}$ , i.e.  $\underline{v}t = k\tilde{\underline{v}}\tilde{t} = \tilde{\underline{v}}t$ .

The parallelism in space is invariant under Lorentz transformations, since those are affine. Thus, together with the invariance of the speed of light, the first observation is verified.

Let us consider the situation in the hint. The particle Q has traveled the distance  $c\Delta t$  in between the occurrence of  $\mathcal{P}$  and  $\mathcal{Q}$ . Thus

$$(\mathcal{P} - \mathcal{Q}, \mathcal{P} - \mathcal{Q}) = (c\Delta t)^2 - (\Delta \vec{x})^2 = (c\Delta t)^2 - [(\Delta r)^2 + (\Delta l + c\Delta t)^2]$$
$$= -[(\Delta r)^2 + (\Delta l)^2 + 2c(\Delta l)(\Delta t)].$$

This does not depend on  $\Delta t$ , and hence not on the events  $\mathcal{P}$ ,  $\mathcal{Q}$ , iff  $\Delta l = 0$ . Since  $(\mathcal{P} - \mathcal{Q}, \mathcal{P} - \mathcal{Q})$  is Lorentz invariant, the property  $\Delta l = 0$  (light particles fly side by side) is invariant as well. In this case the scalar product equals  $-(\Delta r)^2$ .

We thus have shown that O and O' get the same picture. In the case of an object moving w.r.t. O, but at rest w.r.t. O', it follows that O does not see a deformed picture.