Solution 8

1. Calculating with commutators

(i) The equations follow by writing out the commutators.

(ii) By the linearity of the commutator, it is enough to show the first statement for monomials: $f(X) = X^n$, $f'(X) = nX^{n-1}$. In this case the statement holds true for $n = 1$ ($(\mathrm{i}/\hbar)[P, X] = 1$). For $n > 1$ we use induction: by (i), we have

$$\frac{\mathrm{i}}{\hbar}[P, X^n] = \frac{\mathrm{i}}{\hbar}[P, X]X^{n-1} + X\frac{\mathrm{i}}{\hbar}[P, X^{n-1}] = X^{n-1} + X \cdot (n-1)X^{n-2} = nX^{n-1}\,.$$

The second statement is analogous.

(iii) We have $[X_i, X_j] = [P_i, P_j] = 0$, $[X_i, P_j] = \mathrm{i}\hbar\delta_{ij}$ $(i, j = 1, 2, 3)$, $[A, B] = -[B, A]$ and $[A, BC] = B[A, C] + [A, B]C$. Hence

$$\begin{aligned}
[L_{i+1}, L_{i+2}] &= [X_{i+2}P_i - X_iP_{i+2}, X_iP_{i+1} - X_{i+1}P_i] \\
&= X_{i+2}[P_i, X_i]P_{i+1} + X_{i+1}[X_i, P_i]P_{i+2} \\
&= \mathrm{i}\hbar(X_{i+1}P_{i+2} - X_{i+2}P_{i+1}) \\
&= \mathrm{i}\hbar L_i\,, \qquad (i = 1, 2, 3 \mod 3)\,.
\end{aligned}$$

Let $L_3\psi = m\psi$. Then

$$\mathrm{i}\hbar\langle\psi, L_1\psi\rangle = \langle\psi, L_2L_3 - L_3L_2\psi\rangle = (m - m)\langle\psi, L_2\psi\rangle = 0$$

and similarly $\langle\psi, L_2\psi\rangle = 0$.

2. Moving on a line and tunnelling

The time-independent Schrödinger equation (1) in the three areas is

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0\,, \qquad (\mathrm{I} + \mathrm{III})$$

$$\frac{d^2\psi}{dx^2} + \frac{2m(E - V_0)}{\hbar^2}\psi = 0\,. \qquad (\mathrm{II})$$

There it is solved separately by

$$\begin{aligned}
\psi_{\mathrm{I}}(x) &= A_1\mathrm{e}^{\mathrm{i}kx} + B_1\mathrm{e}^{-\mathrm{i}kx}\,, \\
\psi_{\mathrm{II}}(x) &= A_2\mathrm{e}^{\mathrm{i}lx} + B_2\mathrm{e}^{-\mathrm{i}lx}\,, \\
\psi_{\mathrm{III}}(x) &= A_3\mathrm{e}^{\mathrm{i}kx}\,,
\end{aligned}$$

with $k^2 = 2mE/\hbar^2$ and $l^2 = 2m(E - V_0)/\hbar^2$. (In case (a), l is purely imaginary). Furthermore, integrating the Schrödinger equation over $[x_0 - \varepsilon, x_0 + \varepsilon]$, $\varepsilon \to 0$, yields

$$\frac{d\psi}{dx}(x_0+) - \frac{d\psi}{dx}(x_0-) = \lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\frac{d^2\psi}{dx^2}(x)dx = -\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}(E - V(x))\psi(x)dx = 0\,. \quad (3)$$

Thus $d\psi/dx$, and therefore also ψ, are continuous at each position x_0. The last equation in (3) follows from the integrand being bounded in each point of the interval $[x_0-\varepsilon, x_0+\varepsilon]$. Hence for $x_0 = \pm a/2$, this yields

$$\begin{aligned}
A_1 e^{-\frac{ika}{2}} + B_1 e^{\frac{ika}{2}} &= A_2 e^{-\frac{ila}{2}} + B_2 e^{\frac{ila}{2}} \,, \\
ikA_1 e^{-\frac{ika}{2}} - ikB_1 e^{\frac{ika}{2}} &= ilA_2 e^{-\frac{ila}{2}} - ılB_2 e^{\frac{ila}{2}} \,, \\
A_2 e^{\frac{ila}{2}} + B_2 e^{-\frac{ila}{2}} &= A_3 e^{\frac{ika}{2}} \,, \\
ilA_2 e^{\frac{ila}{2}} - ilB_2 e^{-\frac{ila}{2}} &= ikA_3 e^{\frac{ıka}{2}} \,.
\end{aligned}$$

From the last two equations we get

$$A_2 = \frac{k+l}{2l} e^{i(k-l)\frac{a}{2}} A_3 \,, \qquad B_2 = -\frac{k-l}{2l} e^{i(k+l)\frac{a}{2}} A_3 \,,$$

and hence the first two equations imply

$$A_1 = [(k+l)^2 e^{-ila} - (k-l)^2 e^{ila}] \frac{e^{ika}}{4kl} A_3 \,, \qquad B_1 = \frac{k^2-l^2}{4kl} (e^{-ila} - e^{ila}) A_3 \,.$$

A wave $\psi(x) = Ae^{ikx}$ has current density

$$j(x) = \frac{\hbar}{m} \mathrm{Im} \overline{\psi(x)} \psi'(x) = \frac{\hbar k}{m} |A|^2 \,,$$

and therefore $T = |A_3/A_1|^2$ and $R = |B_1/A_1|^2$. Thus

$$T = \frac{|16k^2 l^2|}{D} \,, \qquad R = \frac{|k^2-l^2|^2 |e^{-ila} - e^{ila}|^2}{D} \,,$$

with $D = |(k+l)^2 e^{-ila} - (k-l)^2 e^{ila}|^2$.

(a) $0 < E < V_0$: We write $l = i\lambda$ with $\lambda^2 = 2m(V_0 - E)/\hbar^2 > 0$. The denominator D is

$$\begin{aligned}
D &= |(k+i\lambda)^2 e^{\lambda a} - (k-i\lambda)^2 e^{-\lambda a}|^2 \\
&= (k^2-\lambda^2)(e^{\lambda a} - e^{-\lambda a})^2 + (2k\lambda)^2 (e^{\lambda a} + e^{-\lambda a})^2 \\
&= 4[(k^2+\lambda^2)^2 \sinh^2 \lambda a + 4k^2\lambda^2] \,,
\end{aligned}$$

and we have $T = 16k^2\lambda^2/D$ and $R = 4(k^2+\lambda^2)^2(\sinh^2 \lambda a)/D$; thus $T + R = 1$. It is therefore enough to discuss T:

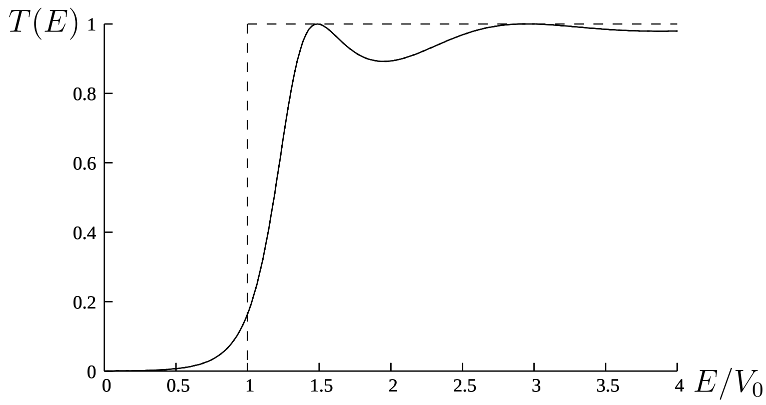
$$T(E) = \frac{4E(V_0-E)}{4E(V_0-E) + V_0^2 \sinh^2\left(\sqrt{\frac{2m(V_0-E)}{\hbar^2}}\, a\right)} \,,$$

and in particular

$$\lim_{E\searrow 0} T(E) = 0 \,, \qquad \lim_{E\nearrow V_0} T(E) = \lim_{E\nearrow V_0} \frac{4E}{4E + \frac{2ma^2}{\hbar^2} V_0^2} = \frac{1}{1 + \frac{ma^2 V_0}{2\hbar^2}} \,.$$

(b) $0 < V_0 < E$:

$$\begin{aligned}
D &= [(k+l)^2 - (k-l)^2]^2 \cos^2 la + [(k+l)^2 + (k-l)^2]^2 \sin^2 la \\
&= 4[4k^2l^2 + (k^2-l^2)^2 \sin^2 la] \,,
\end{aligned}$$



and $T = 16k^2l^2/D$, $R = 4(k^2 - l^2)^2(\sin^2 la)/D$; thus $T + R = 1$.

$$T(E) = \frac{4E(E - V_0)}{4E(E - V_0) + V_0^2 \sin^2\left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}a\right)} \,.$$

Moreover, $T(E) = 1$ for $\sqrt{\frac{2m(E-V_0)}{\hbar^2}}a = n\pi$, i.e. $E = V_0 + \frac{\hbar^2\pi^2}{2ma^2}n^2$, $(n = 1, 2, \ldots)$.

For $\hbar \to 0$ the tunnelling vanishes: $T(E) \to 0$ $(0 < E < V_0)$. However, for $E > V_0$ we do not get $T(E) \to 1$ in that limit; $T(E)$ rather oscillates faster and faster between 1 and

$$\frac{4E(E - V_0)}{4E(E - V_0) + V_0^2} = \frac{4E(E - V_0)}{(2E - V_0)^2}$$

in that case. The discrepancy between this result and the classical one is explained by the following consideration: if the wave length $\lambda = 2\pi/k$ is much smaller compared to the other relevant length scales of the problem, wave optics passes into ray optics and the problem becomes a classical one. However in the present problem, even though $\lambda \to 0$, $(\hbar \to 0)$, there is a length scale w.r.t. which λ is not small, namely the length in which the potential changes by a significant amount; the latter is zero because of the incontinuity of $V(x)$.

3. Current and momentum

We have $\psi'(x) = \mathrm{i}(k_1a_1\mathrm{e}^{\mathrm{i}k_1x} + k_2a_2\mathrm{e}^{\mathrm{i}k_2x})$, and hence

$$\overline{\psi(0)}\psi'(0) = \sum_{i,j=1}^{2} \mathrm{i}\bar{a}_ik_ja_j \,,$$

$$j(0) = \sum_{i,j=1}^{2} (k_i + k_j)\bar{a}_ia_j \,.$$

This is a quadratic form in a_1, a_2 which is not positive semidefinite, since its determinant is

$$\begin{vmatrix} 2k_1 & k_1 + k_2 \\ k_1 + k_2 & 2k_2 \end{vmatrix} = 4k_1k_2 - (k_1 + k_2)^2 = -(k_1 - k_2)^2 < 0 \,.$$