Solution 11

1. Half a harmonic oscillator

The unitary operator $(U\psi)(x) = \psi(-x)$ on $L^2(\mathbb{R})$ corresponds to the reflection in space $\mathbb{R} \to \mathbb{R}$, $x \mapsto -x$. Being unitary, the eigenvalues $\lambda \in \mathbb{C}$ of U satisfy $|\lambda| = 1$. Moreover, since U is also an involution, it is a self-adjoint operator and hence its eigenvalues are ± 1. The two eigenspaces $E_\pm = \{\psi \in L^2(\mathbb{R}) \mid \psi(-x) = \pm\psi(x)\}$ consist of the even and odd wave functions respectively. The Hamiltonian H of the (full) harmonic oscillator commutes with U, i.e. $[H, U] = 0$; hence H leaves $E_\pm$ invariant and the eigenstates of H are either even or odd. Indeed,

$$\psi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} H_n(x) \mathrm{e}^{-x^2/2} = (-1)^n \psi_n(-x)\,,$$

as seen from $H_n(x) = \mathrm{e}^{x^2}(-d/dx)^n \mathrm{e}^{-x^2}$. The eigenvalues are $E_n = n + 1/2$ in our units. The ψ_n with n odd satisfy $\psi_n(x = 0) = 0$, and hence their restrictions to $x \geqslant 0$ are eigenstates of half of the harmonic oscillator (after multiplication with $\sqrt{2}$ in order to recover the normalization). Conversely the continuation as an odd function from $[0, \infty)$ to $\mathbb{R}$ of an eigenstate of half of the harmonic oscillator is an eigenstate of the full oscillator, since $\psi'(x)$ is then continuous at $x = 0$.

2. Time-energy uncertainty principle

(i) The definition of $\dot{A}$ together with (2) yields

$$|\dot{A}| = \Big|\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle_{\psi_t}\Big| = \Big|\frac{\mathrm{d}}{\mathrm{d}t}\langle A(t)\rangle_{\psi}\Big| = \frac{1}{\hbar}\big|\langle [H, A(t)]\rangle_\psi\big| = \frac{1}{\hbar}\big|\langle [H, A]\rangle_{\psi_t}\big| \leqslant \frac{2}{\hbar}\Delta H \cdot \Delta A\,,$$

where the fourth equality holds true since H commutes with the time evolution; and the inequality in the last step follows using the uncertainty principle. We therefore have

$$\Delta H \cdot \Delta t \equiv \Delta H \cdot \frac{\Delta A}{|\dot{A}|} \geqslant \frac{\hbar}{2}\,.$$

(ii) For $f(t) = |\langle \psi_0, \psi_t\rangle|^2 = \langle \psi_t, \psi_0\rangle\langle\psi_0, \psi_t\rangle$ we have

$$\dot{f}(t) = \frac{\mathrm{i}}{\hbar}\big(\langle\psi_t, H\psi_0\rangle\langle\psi_0, \psi_t\rangle - \langle\psi_t, \psi_0\rangle\langle\psi_0, H\psi_t\rangle\big) = \frac{\mathrm{i}}{\hbar}\langle\psi_t, (HP - PH)\psi_t\rangle\,,$$

where $P = \psi_0\psi_0^*$. Thus $|\dot{f}(t)| \leqslant (2/\hbar)\Delta H \cdot \Delta P$ by the uncertainty principle. Moreover we have

$$\Delta P = \sqrt{\langle P^2\rangle_{\psi_t} - \langle P\rangle_{\psi_t}^2} = \sqrt{\langle P\rangle_{\psi_t}(1 - \langle P\rangle_{\psi_t})}\,.$$

Since $\langle P\rangle_{\psi_t} = f(t)$ we get

$$|\dot{f}(t)| \leqslant \frac{2\Delta H}{\hbar}\sqrt{f(t)(1 - f(t))}\,,$$

(where ΔH is independent of t, since H commutes with the time evolution). In particular, the following differential inequality holds true:

$$\dot{f} \geqslant -\frac{2\Delta H}{\hbar}\sqrt{f(1-f)}$$

with $f(0) = 1$. The solution of the corresponding differential equation

$$\dot{f}_0 = -\frac{2\Delta H}{\hbar}\sqrt{f_0(1-f_0)}$$

with $f_0 = 1$ is $f_0(t) = \cos^2(\Delta H \cdot t/\hbar)$. (Use

$$\int \frac{df_0}{\sqrt{f_0(1-f_0)}} = -2\arccos\sqrt{f_0} + C\,,$$

or plug in.) Using (4) we get $f(t) \geqslant \cos^2(\Delta H \cdot t/\hbar)$, and hence

$$\cos^2(\Delta H \cdot t/\hbar) \leqslant |\langle \psi_0, \psi_t \rangle|^2\,. \tag{5}$$

For $t = t_0$, (5) becomes

$$\cos^2(\Delta H \cdot t_0/\hbar) \leqslant 0\,,$$

and positivity of $\cos^2$ yields $\Delta H \cdot t_0 = \hbar n\pi/2$, which implies desired statement.

(iii) The state shifted by x, $\psi_x(x') = \psi_0(x' - x)$, satisfies the differential equation $(d/dx)\psi_x = -\psi'_x$, i.e. (3) with $H = p$ and $t = x$.