Exercise 1. A simple congruential random number generator

Goal: In this exercise we are going to implement a basic congruential random number generator (proposed by Lehmer in 1948).

Write a program that generates random numbers according to

\[ x_i = (cx_{i-1}) \mod p. \]

At first, consider \( c = 3 \) and \( p = 31 \).

**Task 1:** Check your generated random numbers for correlations using the square test. Plot \( x_i \) vs \( x_{i+1} \).

**Task 2:** Repeat the same using the cube test. Plot \( x_i \) vs \( x_{i+1} \) vs \( x_{i+2} \).

**Task 3:** Repeat the same for a different random number generator. This can be achieved e.g. by choosing a different \( c \) and \( p \).

**Task 4:** [Optional] Check the built-in random number generators your programming language provides.

Exercise 2. The \( \chi^2 \)-test

Goal: Here, we are getting to know to another random number generator test.

**Task:** Test your random number generators from exercise 1 using the \( \chi^2 \)-test.

The \( \chi^2 \)-test is described in the following:

- Divide the range of random numbers into \( k \) bins i.e. discrete intervals of the same size such that the probability of a random number to be in the interval \( i \) is given by \( p_i = 1/k \).

- Using each random number generator, generate at least one sequence of \( n \) numbers. For each sequence, measure the count \( N_i \) of random numbers in each interval \( i \).

- Compute the \( \chi^2 \)-value for one specific sequence of random numbers

\[
\chi^2 = \sum_{i=1}^{k} \frac{(N_i - np_i)^2}{np_i}.
\]

- Use the table chi_square_description.pdf (from Donald E. Knuth, The Art of Computer Programming, Volume 2) to check the reliability of your random number generators.

Exercise 3. Random numbers on a circle

Goal: Finally, we are learning how random numbers are distributed uniformly on a circle.

**Task:** Consider a circle of radius \( R \). Generate a homogeneous distribution of random points within the circle.
Hint: If we draw \((r, \varphi)\) uniformly in \([0, R] \times [0, 2\pi]\), the points are not homogeneously distributed within the circle!