The excess spin and edge representations of symmetries in ground state phases

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Ground states of quantum spin systems

- Quantum spin system: a countable collection of finite dimensional quantum systems, labeled by $x \in \Gamma$
- For finite $\Lambda \subset \Gamma$: local Hamiltonian

$$H^{\Lambda} = \sum_{X \subset \Lambda} \Phi(X), \qquad \Phi(X) = \Phi(X)^* \in \mathcal{A}_X$$

  with fast decay of $\Phi(X)$ in the size of X
- Our interest: ground state spaces $\mathcal{S}^{\Lambda}$ of H^{Λ}, their limit $\mathcal{S}^{\Gamma}$ as $\Lambda \to \Gamma$
- Either: Understand specific models in details,
  or: Identify equivalence classes of models with similar behaviour

Today: Gapped ground state phases of quantum spin chains, local order versus string or topological order

On ‘topological order’

- So far: phases are defined by symmetries and their breaking
  Multiple Gibbs states (or ground states) arise from symmetry breaking
- In particular, different states can be distinguished locally
  In the Ising model, the magnetization in the x direction: $M_i = \omega_i(S^0)$

- Kitaev’s model (2003). Spin model defined on 2-dimensional surfaces
  - when defined on a surface of genus g, the Hamiltonian has 4^g ground states
  - expectation values of local observables are independent of the ground state: no local order parameter

$\rightsquigarrow$ Phases carrying topological order: Locally disordered, topological order parameters

The Haldane phase

An typical example: The Heisenberg antiferromagnet on a graph (Γ, E)

$$H = \sum_{(x,y)\in E} J_{xy} S^x \cdot S^y, \quad J_{xy} > 0.$$

Haldane's conjecture for $\Gamma = \mathbb{Z}$ (1983):

- Half-integer spins: Unique ground state, polynomial decay of correlations, no spectral gap

  Affleck-Lieb (1986), Aizenman-Nachtergaele (1994)

- Integer spins: Unique ground state, exponential decay of correlations, spectral gap

  The Haldane phase exists: $S = 1$ AKLT model (1988)

  Exponential clustering: Spectral gap $\Rightarrow$ exponential decay

  Hastings (2004), Nachtergaele-Sims (2006)

AKLT model: Ground states

$$H^{[a,b]} = \sum_{x=a}^{b-1} \left[\frac{1}{2} \left(S^x \cdot S^{x+1} \right) + \frac{1}{6} \left(S^x \cdot S^{x+1} \right)^2 + \frac{1}{3} \right] = \sum_{x=a}^{b-1} P_{(2)}^{x,x+1}$$

- Uniform spectral gap above the ground state energy $\gamma^{[a,b]} > 0.137194$
- Unique ground state ω on $\mathbb{Z}$.
- No long range order: $|\omega(S_3^0 S_3^x)| \propto 3^{-x}$
- 'Dilute Néel order': In the product basis $(-, 0, +)$ of all S_3^x, ω is a superposition of all words of the form

$$\cdots + 00000 - 000 + -0 + 000000 - 0 + 000 - \cdots$$

- String order: den Nijs-Rommelse (1989), Kennedy-Tasaki (1992)

$$\lim_{x \to \infty} \left| \omega \left(S_3^0 \mathrm{e}^{\mathrm{i}\pi \sum_{j=1}^{x-1} S_3^j} S_3^x \right) \right| = C > 0$$

AKLT model: Excess spin

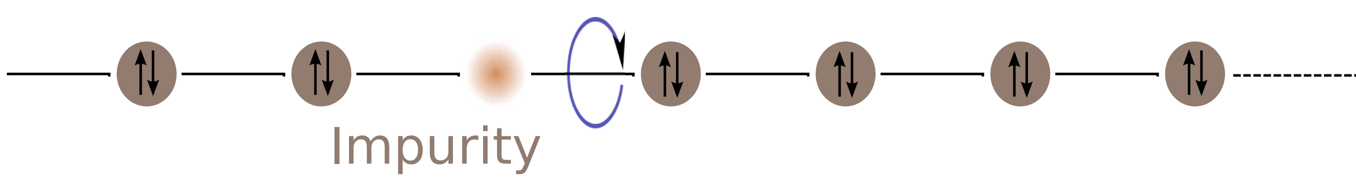
All related properties to string order:

- Degenerate ground state on half-infinite chains: edge states
- Edge state space is isomorphic to a Bloch sphere
- Existence of an excess spin:
  Representation of $SU(2)$ in the GNS Hilbert space of ω

$$U_g = \operatorname*{s-lim}_{L\to\infty} \exp\left(\mathrm{i} g \cdot \sum_{x=1}^{L} \pi_\omega(S^x) \right)$$

In fact, fractional spin at the edge is a characteristics of the Haldane phase

Experimental measurement: Hagiwara et al (1990)



What is a phase?

Vaguely: An equivalence class of qualitatively similar models

No phase transition without closing the gap (Ising in transverse field)

Assume:

- The spectral gap above the ground state energy of

$$H_{\Lambda_n}(s) = \sum_{X \subset \Lambda_n} \Phi_s(X), \qquad s \in [0,1],$$

  is uniformly bounded below by $\gamma > 0$, in n and s.

- Φ_s is smooth, short range for all s

Let $\mathcal{S}(s)$ be the set of weak-* limits of ground states of $H_{\Lambda_n}(s)$. E.g., if there is only one ground state: for each local observable A,

$$\omega_s(A) = \lim_{\Lambda_n \to \Gamma} \langle \psi_{\Lambda_n}(s), A\psi_{\Lambda_n}(s) \rangle$$

Equivalence

Theorem. [B, Michalakis, Nachtergaele, Sims] *There exists a cocycle of automorphisms* $\alpha_{s,s'}$ *of the algebra of observables such that*

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_{s,0}$$

for $s \in [0, 1]$. *The automorphisms* $\alpha_{s,s'}$ *can be constructed as the thermodynamic limit of the* s*-dependent evolution for an interaction* $\Psi(X, s)$, *which decays almost exponentially.*

Concretely, the action of the quasi-local transformations $\alpha_s = \alpha_{s,0}$ on observables is given by $\alpha_s(A) = \lim_{n\to\infty} V_n^*(s) A V_n(s)$, where $V_n(0) = 1$ and $V_n(s)$ solves a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = \mathrm{i} D_n(s) V_n(s), \qquad \text{with} \qquad D_n(s) = \sum_{X \subset \Lambda_n} \Psi(X, s)$$

Picture of a phase

- If there exists a gapped path $H(s)$ between H_0 and H_1, then their ground state spaces are globally equivalent
- Locality of $\alpha_{s,s'}$:

$$\|[\alpha_{s,s'}(A), B]\| \leq \|A\|\|B\|F\Big(d(\mathrm{supp}(A), \mathrm{supp}(B))\Big)$$

  where F decays faster than any polynomial
- True on any lattice: $\mathbb{Z}$, but also $[1,\infty)$ or $(-\infty, 0]$
- Precisely: the ground state spaces $\mathcal{S}^\Gamma(s)$ are homeomorphic
  In particular: same structure of edge states
- Symmetries? $\alpha_{s,s'}$ has all symmetries of the Hamiltonians:

$$\theta_g(\Phi_s(X)) = \Phi_s(X) \qquad \Longrightarrow \qquad \alpha_{s,s'} \circ \theta_g = \theta_g \circ \alpha_{s,s'}$$

Abstract excess spin

The excess spin arises as a representation of rotations on the half-line.

Natural question: is it constant within a phase?

For $g \in SU(2)$, ρ a state, and $A \in \mathcal{A}^{[1,\infty)}$

$$\Theta_g^{[1,\infty)}(\rho)(A) := \rho(\theta_g^{[1,\infty)}(A))$$

Theorem. [B, Nachtergaele] *Let $H(s)$ be a path of gapped $SU(2)$ invariant Hamiltonians, and let*

$$\Theta_{s,g}^{[1,\infty)} = \Theta_g^{[1,\infty)} \restriction_{\mathcal{S}^{[1,\infty)}(s)}$$

Then the representations $\Theta_{s,\cdot}^{[1,\infty)}$ are all equivalent

I.e. models in the same symmetric phase carry equivalent representations of the symmetry group on the edge ground state spaces

Proof

i. $SU(2)$-invariant interaction
$\Longrightarrow \mathcal{S}^{[1,\infty)}(s)$ is $SU(2)$-invariant
$\Longrightarrow \Theta_{s,\cdot}^{[1,\infty)}$ is a subrepresentation

ii. The theorem follows from

$$\omega^{[1,\infty)}\big(\alpha_s(\theta_g(A))\big) = \omega^{[1,\infty)}\big(\theta_g(\alpha_s(A))\big)$$

i.e.

$$\Theta_{s,g} = \alpha_s^* \circ \Theta_{0,g} \circ (\alpha_s^*)^{-1}$$

and α_s^* is an isomorphism.

Note: the specific form of α_s is essential

Existence of the excess spin I

- The generator of these rotations $\sum_{x=1}^{\infty} S^x$ does not exist in algebra
- But: if spin-spin correlation in the ground state decay fast enough: may exist in the GNS representation π_ω

  $\rightsquigarrow$ unitary implementation of the rotations
- Study limit of regularized sum

$$\pi_\omega\left(S^+(\epsilon)\right) := \sum_{x=1}^{\infty} \mathrm{e}^{-\epsilon x} \pi_\omega\left(S^x\right)$$

We can prove the existence of the limit in two cases

- For models with finitely correlated ground states (matrix product states)
- For models that have a stochastic-geometric representation

The stochastic picture

Example (Aizenman-Nachtergaele 1994): Heisenberg antiferromagnet:

$$H = \sum_{e:\text{edges}} \left[(1/2) - P^e_{(0)}\right]$$

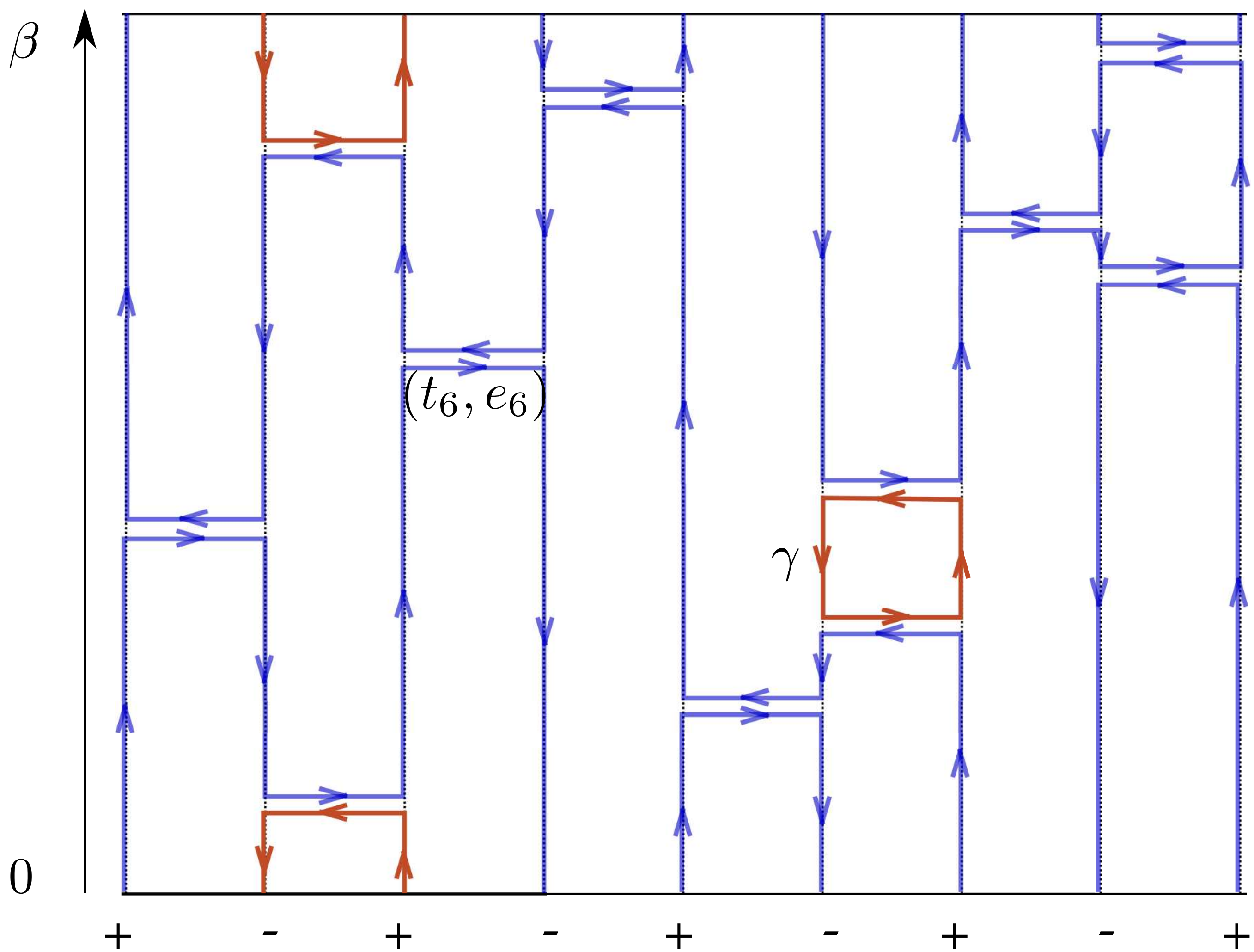
Duhamel expansion $\quad \longleftrightarrow \quad$ $2d$ classical random loop model

$$\mathrm{Tr}(\mathrm{e}^{-\beta H}) = \sum_{k=0}^{\infty} \int_0^{\beta} dt_1 \cdots \int_0^{t_{k-1}} dt_k \mathrm{e}^{-\beta L/2} \sum_{e_1 \ldots e_k} \mathrm{Tr}\left(P^{e_1}_{(0)} \cdots P^{e_k}_{(0)}\right)$$

$$= \int d\rho(\nu) 2^{\#\text{ loops}} = \int d\mu(\nu)$$

- $d\rho(\nu)$ is a Poisson process of bridges (t_i, e_i), rate 1, up to time β
- $P^{e_i}_{(0)}$ 'sits' on the bridge (t_i, e_i)
- Bridges determine a loop configuration
- Each loop configuration determines $\left\langle \sigma_0, P^{e_1}_{(0)} \sigma_1 \right\rangle \cdots \left\langle \sigma_{k-1}, P^{e_k}_{(0)} \sigma_0 \right\rangle$

Classical loop configurations



The stochastic picture

- The ground state is obtained for the stochastic model on the plane
- Crucially,

$$\omega(S^x S^y) \propto \mathbb{P}_\mu(\text{there is a loop between } x \text{ and } y)$$

$\rightsquigarrow$ Loops express physical correlations

Expect: For each configuration ν, with loops $\gamma \in \nu$

$$\sum_{x \in \gamma : \gamma \text{ intersects } (-\infty, 0]} S^x$$

contribute to $\sum_{x=1}^{\infty} \pi_\omega(S^x)$, as loops closed in $[1, \infty)$ sum to 0.
Therefore: Excess spin exists if large loops are rare, i.e. with sufficient decay of correlations

Which models?

Spin-$1/2$ Heisenberg antiferromagnet and higher spin generalizations

i. Represent a spin S as the highest spin component – totally symmetric subspace – of $2S$ spins $1/2$

ii. Interaction:

$$V^* \prod_{l=1}^{k} \left[(1/2) - P_{(0)}^{(x,l),(x+1,l)} \right] V$$

where $V\mathcal{D}^S = (\mathcal{D}^{1/2})^{\otimes 2S} V$

Defines a 'multiline' model

Also expressed as polynomials in $S^x \cdot S^y$

$$H = -\sum_x \sum_{k=0}^{2S} J_k Q_k (S^x \cdot S^{x+1})$$

Existence of the excess spin

Theorem. [B, Nachtergaele] *Let H be as above. If*

$$\sum_{x\in\mathbb{Z}} \left| x^3\, \omega(S^0 S^x) \right| < \infty,$$

then the strong limit

$$U_g^+ = \operatorname*{s-lim}_{\epsilon\longrightarrow 0} \mathrm{e}^{\mathrm{i}g\cdot\pi_\omega\left(S^+(\epsilon)\right)}$$

exists and defines a strongly continuous representation of $SU(2)$
Moreover:

i. *Let S^+ be the self-adjoint generator of U_g^+*

$$S^+ = \operatorname*{s-lim}_{\epsilon\longrightarrow 0} \pi_\omega\left(S^+(\epsilon)\right)$$

ii. $\{\pi_\omega(A)\Omega : A \in \mathcal{A}_{\mathrm{loc}}\}$ *is contained in the domain of S^+*

Idea of proof

In the stochastic picture: ν are the loop configurations and

$$\omega(A) = \int d\mu(\nu) E_\nu(A)$$

for functions $f((S^x)_{x\in\Gamma})$,

$$E_\nu(f) \propto \sum_{\sigma \text{ compatible with } \nu} f(\sigma(0))$$

The theorem follows from

$$\lim_{\max(\epsilon,\epsilon')\to 0} \omega\left(A^* \left(S(\epsilon) - S(\epsilon')\right)^2 B\right) = 0$$

Reduce to the case $A = B = 1$ using decay of correlations: few loops connect $\mathrm{supp}(B)$ to the far right

Then prove

$$\int d\mu(\nu) \left[E_\nu \left(S(\epsilon) - S(\nu) \right)^2 \right] \longrightarrow 0$$

For loops connecting the two half-lines:

$$\begin{aligned}
&\int d\mu(\nu) E_\nu \Big[\Big(\sum_{x \geq 1}{}' (\mathrm{e}^{-\epsilon x} - 1) S^x \Big)^2 \Big] \\
&\qquad \propto \int d\mu(\nu) \sum_{x,y \geq 1} (\mathrm{e}^{-\epsilon x} - 1)(\mathrm{e}^{-\epsilon y} - 1) I_\nu \big[x \sim y \sim (-\infty, 0] \big] \\
&\qquad \propto \sum_{x,y \geq 1} (\mathrm{e}^{-\epsilon x} - 1)(\mathrm{e}^{-\epsilon y} - 1) \mathbb{P}_\mu (x \sim y \sim (-\infty, 0])
\end{aligned}$$

conclude by $\mathbb{P}_\mu(x \sim y) \propto \omega(S^0 S^x)$ and dominated convergence
Similar for the loops remaining in the positive half-line

Conclusion

- Gapped ground state phase as a set of models that can be smoothly related without closing the spectral gap
- Implies the 'local unitary' equivalence of ground state spaces
- For phases with a symmetry group G: equivalent representations of G at the edges
- Unitary implementability of the $SU(2)$ action of on the GNS spaces: Excess spin operator

- General classification of phases
- Topological order
- Long range entanglement, area laws
- Anyons?