Resolvent Algebras

An alternative approach to canonical quantum systems

Detlev Buchholz

"Analytical Aspects of Mathematical Physics"
ETH Zürich May 29, 2013

Motivation

Kinematics of quantum systems [Born, Heisenberg, Jordan]:

X: real (finite or infinite dimensional) vector space
$\sigma : X \times X \to \mathbb{R}$ non–degenerate symplectic form
Consider algebra $\mathcal{P}(X, \sigma)$ generated by basic observables $\phi(f)$, $f \in X$

$$\phi(cf + c'f') = c\,\phi(f) + c'\,\phi(f')\,, \quad \phi(f)^* = \phi(f)\,,$$
$$[\phi(f), \phi(f')] = i\,\sigma(f, f')\,1\,.$$

Difficulties:

- elements of $\mathcal{P}(X, \sigma)$ are (intrinsically) unbounded
- group of *–automorphisms $\mathrm{Aut}\,\mathcal{P}(X, \sigma)$ is small (does not contain interesting dynamics: quadratic Hamiltonians)

Recipe:
Proceed to C*–algebra containing the same algebraic information:

(1) Weyl–algebra $\mathcal{W}(X,\sigma)$; generators $W(f) = e^{i\phi(f)}$, $f \in X$ satisfy

$$W(f)W(f') = e^{i\sigma(f,f')}\, W(f')W(f)\,, \quad W(f)^* = W(-f)$$

Difficulties:

- no interesting dynamics (reason: $\mathcal{W}(X,\sigma)$ simple algebra)
- representation theory not manageable . . .

(2) $\mathcal{C}(\mathcal{H})$ (group algebra generated by $\mathcal{W}(X,\sigma)$)

Difficulties:

- works only for finite systems
- specific features of system disappear ($\phi(f)$ not affiliated) . . .

(3) $\mathcal{B}(\mathcal{H})$ (multiplyer algebra of $\mathcal{C}(\mathcal{H})$)

Difficulties:

- nonspecific input (oversized algebra)
- dynamics for infinite systems (construction of s.a. generators) . . .

(1) Weyl–algebra $\mathcal{W}(X,\sigma)$; generators $W(f) = e^{i\phi(f)}$, $f \in X$ satisfy

$$W(f)W(f') = e^{i\sigma(f,f')}\, W(f')W(f)\,, \quad W(f)^* = W(-f)$$

Difficulties:

- no interesting dynamics (reason: $\mathcal{W}(X,\sigma)$ simple algebra)
- representation theory not manageable ...

(2) $\mathcal{C}(\mathcal{H})$ (group algebra generated by $\mathcal{W}(X,\sigma)$)

Difficulties:

- works only for finite systems
- specific features of system disappear ($\phi(f)$ not affiliated) ...

(3) $\mathcal{B}(\mathcal{H})$ (multiplyer algebra of $\mathcal{C}(\mathcal{H})$)

Difficulties:

- nonspecific input (oversized algebra)
- dynamics for infinite systems (construction of s.a. generators) ...

(1) Weyl–algebra $\mathcal{W}(X,\sigma)$; generators $W(f)=e^{i\phi(f)}$, $f\in X$ satisfy

$$W(f)W(f')=e^{i\sigma(f,f')}\,W(f')W(f)\,,\quad W(f)^*=W(-f)$$

Difficulties:

- no interesting dynamics (reason: $\mathcal{W}(X,\sigma)$ simple algebra)
- representation theory not manageable …

(2) $\mathcal{C}(\mathcal{H})$ (group algebra generated by $\mathcal{W}(X,\sigma)$)

Difficulties:

- works only for finite systems
- specific features of system disappear ($\phi(f)$ not affiliated) …

(3) $\mathcal{B}(\mathcal{H})$ (multiplyer algebra of $\mathcal{C}(\mathcal{H})$)

Difficulties:

- nonspecific input (oversized algebra)
- dynamics for infinite systems (construction of s.a. generators) …

Proposal: [DB, H Grundling]

Proceed to algebra generated by resolvents of the basic observables

$$R(\lambda, f) \doteq (i\lambda 1 + \phi(f))^{-1}, \quad f \in X,\ \lambda \in \mathbb{R}\backslash\{0\}\,.$$

Remark: All algebraic properties of the basic observables can be expressed in terms of polynomial relations between these resolvents. They determine abstractly the

Resolvent Algebra

Outline

- Motivation ✓
- Resolvent algebra
- Basic properties
- Applications
- Conclusions

Resolvent algebra

Defining relations: $f, g \in X,\ \lambda, \mu \in \mathbb{R}\backslash\{0\}$

$$R(\lambda, f) - R(\mu, f) = i(\mu - \lambda)R(\lambda, f)R(\mu, f)$$

$$R(\lambda, f)^* = R(-\lambda, f)$$

$$\big[R(\lambda, f),\ R(\mu, g)\big] = i\sigma(f, g)\, R(\lambda, f)\, R(\mu, g)^2 R(\lambda, f)$$

$$\nu\, R(\nu\lambda,\ \nu f) = R(\lambda, f)$$

$$R(\lambda, f)R(\mu, g) =$$
$$= R(\lambda + \mu,\ f + g)\big[R(\lambda, f) + R(\mu, g) + i\sigma(f, g)R(\lambda, f)^2 R(\mu, g)\big]$$

$$R(\lambda, 0) = \tfrac{1}{i\lambda}\, 1$$

Definition: $\mathcal{R}_0(X, \sigma)$ is the unital *-algebra generated by $\{R(\lambda, f)\}$

Lemma

Denote by $(\pi, \mathcal{H})$ *(cyclic) the representations of* $\mathcal{R}_0(X, \sigma)$*. Then*

$$\|R\| \doteq sup_{(\pi,\mathcal{H})} \|\pi(R)\|_{\mathcal{H}} , \quad R \in \mathcal{R}_0(X, \sigma)$$

exists and defines a C^**–norm.*

Definition: Given (X, σ), the associated resolvent algebra $\mathcal{R}(X, \sigma)$ is the completion of $\mathcal{R}_0(X, \sigma)$ with respect to this norm.

Basic properties

Definition: A representation $(\pi, \mathcal{H})$ of $\mathcal{R}(X, \sigma)$ is said to be regular if there exist self–adjoint generators for all resolvents,

$$\pi(R(\lambda, f)) = (i\lambda 1 + \phi_\pi(f))^{-1}, \quad f \in X,\ \lambda \in \mathbb{R}\backslash\{0\}\,.$$

Proposition

- *Any faithful factorial representation of $\mathcal{R}(X, \sigma)$ is regular*
- *Any regular representation of $\mathcal{R}(X, \sigma)$ is faithful*
- *The regular representations of $\mathcal{R}(X, \sigma)$ are in 1–1 correspondence with the regular representations of $\mathcal{W}(X, \sigma)$. (Yet there is no such correspondence between the non–regular representations.)*

Non–regular representations? (constraints, ideal structure, . . .)

Classification of irreducible representations π of $\mathcal{R}(X,\sigma)$ for $\dim X < \infty$:

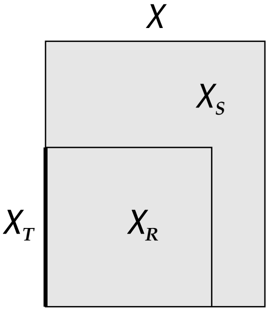


Fig. Decomposition of X for given π

- regular subspace X_r : $\ker \pi(R(\lambda, f)) = \{0\}\,,\ f \in X_r$
- trivial subspace $X_t \subset X_r$: $\pi(R(\lambda, f)) \in \mathbb{C}1 \backslash \{0\}\,,\ f \in X_t$
- singular complement $X_s = X \backslash X_r$: $\pi(R(\lambda, f)) = 0\,,\ f \in X_s$

Finite systems

Proposition

Let (X, σ) be a symplectic space with dim$X < \infty$.

- *$\mathcal{R}(X, \sigma)$ is of type I (postliminal). (Every irreducible representation contains the compact operators)*
- *$\widehat{\pi} \mapsto$ Ker$\widehat{\pi}$ is a bijection from the spectrum $\widehat{\mathcal{R}}(X, \sigma)$ of $\mathcal{R}(X, \sigma)$ to its primitive ideals. (Compare abelian C*–algebras)*
- *The intersection of all closed non–zero ideals of $\mathcal{R}(X, \sigma)$ is isomorphic to $\mathcal{C}(\mathcal{H})$. (Unique minimal ideal)*
- *dim$X$ is a complete algebraic invariant for the resolvent algebras. (Size of system algebraically encoded)*

Infinite systems

Proposition

Let (X, σ) be a symplectic space with dim$X = \infty$.

- $\mathcal{R}(X, \sigma)$ *is the C*–inductive limit of its subalgebras* $\mathcal{R}(Y, \sigma)$*, where* $Y \subset X$ *is finite dimensional and non–degenerate.* *(Each $\mathcal{R}(Y, \sigma)$ contains minimal ideal isomorphic to $\mathcal{C}(\mathcal{H})$; key to the construction of dynamics)*
- $\mathcal{R}(X, \sigma)$ *does not contain any non–zero minimal ideal.*
- $\mathcal{R}(X, \sigma)$ *is nuclear.* *(Unique tensor products)*

Thus the resolvent algebras provide a convenient mathematical framework which encodes specific information about the underlying quantum systems.

Applications

Resolvent algebras have found applications in

- representation theory for abelian Lie algebras of derivations
- study of constraint systems, BRST cohomology
- algebraic framework for SUSY, super KMS functionals
- construction of dynamical systems ...

Technical virtue: algebras accessible to "uniform" and "weak" methods

Applications

Resolvent algebras have found applications in

- representation theory for abelian Lie algebras of derivations
- study of constraint systems, BRST cohomology
- algebraic framework for SUSY, super KMS functionals
- construction of dynamical systems ...

Technical virtue: algebras accessible to “uniform” and “weak” methods

Instructive example: $\mathcal{R}(X, \sigma)$ with $\dim X = 2N$

Recall: Every regular representation is faithful; Schrödinger representation $(\pi_S, \mathcal{H}_S)$

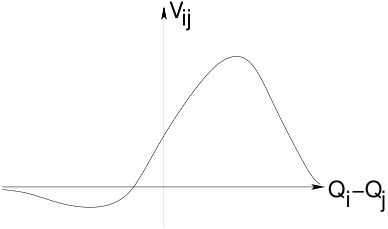
Symplectic basis: $f_i, g_i \in X$; $\quad P_i \doteq \phi_{\pi_S}(f_i)$, $Q_i \doteq \phi_{\pi_S}(g_i)$ $\; i = 1, \dots, N$

Resolvents:

$$(i\lambda 1 + a_1 P_1 + \dots a_N P_N + b_1 Q_1 + \dots b_N Q_N)^{-1} .$$

Standard Hamiltonian: (N interacting particles in one dimension)

$$H = H_0 + V = \sum_i \tfrac{1}{2m_i} P_i^2 + \sum_{i \neq j} V_{ij}(Q_i - Q_j)$$



Proposition

$(i\mu 1 + H)^{-1} \in \pi_S(\mathcal{R}(X,\sigma))$ *for* $\mu \in \mathbb{R}\backslash\{0\}$ *(i.e. H is affiliated with $\mathcal{R}(X,\sigma)$).*

Note: $(i\mu 1 + H)^{-1} \notin \pi_s(\mathcal{W}(X,\sigma)),\ \mathcal{C}(\mathcal{H})$

Sketch of proof:

1. The abelian C*–subalgebra generated by

$$(i\lambda 1 + a_1 P_1 + \dots a_N P_N)^{-1}, \quad a_1, \dots, a_n \in \mathbb{R},\ \lambda \in \mathbb{R}\backslash\{0\}$$

   contains $(i\mu 1 + H_0)^{-1}$, $\mu \in \mathbb{R}\backslash\{0\}$

2. The abelian C*–subalgebra generated by

$$(i\lambda 1 + b_1 Q_1 + \dots b_N Q_N)^{-1}, \quad b_1, \dots, b_n \in \mathbb{R},\ \lambda \in \mathbb{R}\backslash\{0\}$$

   contains $V_{ij}(Q_i - Q_j)$, $i, j = 1, \dots, N$ and hence V

3. The series $(i\mu 1 + H)^{-1} = \sum_{n=0}^{\infty} (i\mu 1 + H_0)^{-1}(V(i\mu 1 + H_0)^{-1})^n$ converges in norm for $|\mu| > \|V\|$ **QED**

Proposition

$(i\mu 1 + H)^{-1} \in \pi_S(\mathcal{R}(X,\sigma))$ *for* $\mu \in \mathbb{R}\backslash\{0\}$ *(i.e. H is affiliated with* $\mathcal{R}(X,\sigma)$*).*

Note: $(i\mu 1 + H)^{-1} \notin \pi_s(\mathcal{W}(X,\sigma)),\ \mathcal{C}(\mathcal{H})$

Sketch of proof:

1. The abelian C*–subalgebra generated by
$$(i\lambda 1 + a_1 P_1 + \dots a_N P_N)^{-1}, \quad a_1, \dots, a_n \in \mathbb{R},\ \lambda \in \mathbb{R}\backslash\{0\}$$
contains $(i\mu 1 + H_0)^{-1}$, $\mu \in \mathbb{R}\backslash\{0\}$
2. The abelian C*–subalgebra generated by
$$(i\lambda 1 + b_1 Q_1 + \dots b_N Q_N)^{-1}, \quad b_1, \dots, b_n \in \mathbb{R},\ \lambda \in \mathbb{R}\backslash\{0\}$$
contains $V_{ij}(Q_i - Q_j)$, $i, j = 1, \dots, N$ and hence V
3. The series $(i\mu 1 + H)^{-1} = \sum_{n=0}^{\infty} (i\mu 1 + H_0)^{-1}(V(i\mu 1 + H_0)^{-1})^n$ converges in norm for $|\mu| > \|V\|$ **QED**

Proposition

$Ad\, e^{itH}\left(\pi_S(\mathcal{R}(X,\sigma))\right) = \pi_S(\mathcal{R}(X,\sigma))$ *(i.e. H generator of dynamics of $\mathcal{R}(X,\sigma)$).*

Sketch of proof:

1. $\mathrm{Ad}\, e^{itH_0}\left(\pi_S(\mathcal{R}(X,\sigma))\right) = \pi_S(\mathcal{R}(X,\sigma))$ (symplectic transformation)
2. Dyson series (norm convergent)

$$e^{itH} e^{-itH_0} = \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, V(t_n) \cdots V(t_1)\,,$$

$V(s) \doteq \mathrm{Ad}\, e^{isH_0}(V) = \sum_{i \neq j} V_{ij}(s) = \sum_{i \neq j} V_{ij}\left((Q_i - Q_j) + s(\frac{1}{m_i}P_i - \frac{1}{m_j}P_j)\right)$

Warning: Integrals only defined in the strong operator topology

3. Let $Y_{i,j} \subset X$ be the subspace corresponding to $(\frac{1}{m_i}P_i - \frac{1}{m_j}P_j), (Q_i - Q_j)$. Then

$$s_1, s_2 \mapsto V_{ij}(s_1) V_{ij}(s_2) \in \pi_S(\mathcal{R}(Y_{i,j},\sigma))\,;$$

for $s_1 \neq s_2$ elements of the compact ideal $\mathcal{C}_{ij} \subset \pi_S(\mathcal{R}(Y_{i,j},\sigma))$

4. $\left(\int_0^t ds V_{ij}(s)\right)^2 = \int_0^t ds_1 \int_0^t ds_2\, V_{ij}(s_1) V_{ij}(s_2) \in \mathcal{C}_{ij}$, hence $\int_0^t ds V_{ij}(s) \in \pi_S(\mathcal{R}(X,\sigma))$
5. Induction argument: $e^{itH} e^{-itH_0} \in \pi_S(\mathcal{R}(X,\sigma))$ QED

Proposition

$Ad\, e^{itH}\big(\pi_s(\mathcal{R}(X,\sigma))\big) = \pi_s(\mathcal{R}(X,\sigma))$ *(i.e. H generator of dynamics of $\mathcal{R}(X,\sigma)$).*

Sketch of proof:

1. $\mathrm{Ad}\, e^{itH_0}\big(\pi_s(\mathcal{R}(X,\sigma))\big) = \pi_s(\mathcal{R}(X,\sigma))$ (symplectic transformation)
2. Dyson series (norm convergent)

$$e^{itH} e^{-itH_0} = \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, V(t_n) \cdots V(t_1)\,,$$

$V(s) \doteq \mathrm{Ad}\, e^{isH_0}(V) = \sum_{i\neq j} V_{ij}(s) = \sum_{i\neq j} V_{ij}\big((Q_i - Q_j) + s(\frac{1}{m_i}P_i - \frac{1}{m_j}P_j)\big)$

Warning: Integrals only defined in the strong operator topology

3. Let $Y_{i,j} \subset X$ be the subspace corresponding to $(\frac{1}{m_i}P_i - \frac{1}{m_j}P_j), (Q_i - Q_j)$. Then

$$s_1, s_2 \mapsto V_{ij}(s_1) V_{ij}(s_2) \in \pi_S(\mathcal{R}(Y_{i,j},\sigma))\,;$$

for $s_1 \neq s_2$ elements of the compact ideal $\mathcal{C}_{ij} \subset \pi_S(\mathcal{R}(Y_{i,j},\sigma))$

4. $\big(\int_0^t ds V_{ij}(s)\big)^2 = \int_0^t ds_1 \int_0^t ds_2\, V_{ij}(s_1) V_{ij}(s_2) \in \mathcal{C}_{ij}$, hence $\int_0^t ds V_{ij}(s) \in \pi_S(\mathcal{R}(X,\sigma))$
5. Induction argument: $e^{itH} e^{-itH_0} \in \pi_s(\mathcal{R}(X,\sigma))$ **QED**

Systems with **infinite** number of degrees of freedom (Example):

vibrating atoms on an infinite lattice; (X, σ) has countable basis

Fock representation of $\mathcal{R}(X, \sigma)$ faithful, local Hamiltonians

$$H_\Lambda := \sum_{j \in \Lambda} \left(\tfrac{1}{2m} P_j^2 + \tfrac{m\,\omega^2}{2} Q_j^2 \right) + \sum_{j, j+1 \in \Lambda} V(Q_j - Q_{j+1}), \quad \Lambda \subset \mathbb{Z};$$

corresponding unitary groups

$$U_\Lambda(t) := e^{itH_\Lambda}, \quad t \in \mathbb{R}$$

Proposition

Let $V \in \mathcal{C}_0$*.*

- H_Λ *is affiliated with* $\mathcal{R}(X, \sigma)$*,* $\Lambda \subset \mathbb{Z}$
- U_Λ *induces automorphic action* $\alpha_\Lambda : \mathbb{R} \to Aut\,\mathcal{R}(X, \sigma)$*,* $\Lambda \subset \mathbb{Z}$
- $\alpha_t = \lim_{\Lambda \nearrow \mathbb{Z}} \ \alpha_{\Lambda\, t}$ *exists on* $\mathcal{R}(X, \sigma)$ *in the strong topology,* $t \in \mathbb{R}$

Straightforward application of C*–algebraic methods:

Proposition

There exists a **regular** *ground state* ω_0 *for* $(\mathcal{R}(X, \sigma), \alpha_\mathbb{R})$*, i.e. in GNS–representation* $(\pi_0, \mathcal{H}_0, \Omega_0)$ *there is* $U_0 : \mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H}_0)$ *with positive generator such that*

$$U_0(t)\, \pi_0(R)\, \Omega_0 = \pi_0(\alpha_t(R))\, \Omega_0\,, \quad R \in \mathcal{R}(X, \sigma),\ t \in \mathbb{R}\,.$$

Concluding remarks

Resolvent algebras

- encode specific information about quantum systems
- have comfortable algebraic properties
- have a manageable representation theory
- include physically relevant observables
- are stable under a wealth of interesting dynamics
- cover infinite systems (Bosonic lattice theories, Pauli–Fierz models?, ...)
- simplify discussion of constraints
- provide a framework for studies of SUSY

are a convenient analytical framework for quantum physics

References:

- DB and HG: The resolvent algebra: A new approach to canonical quantum systems
  Journal of Functional Analysis 254 (2008) 2725
- DB and HG: The resolvent algebra: Ideals and dimension
  arXiv:130x.xxxx
- DB and HG: Lie Algebras of Derivations and Resolvent Algebras
  Comm. Math. Phys. DOI 10.1007/s00220-012-1567-z
- P. Costello: The mathematics of the BRST-constraint method
  arXiv:0905.3570
- DB and HG: Algebraic Supersymmetry: A case study
  Commun. Math. Phys. 272 (2007) 699