Integrable particle systems and Macdonald processes

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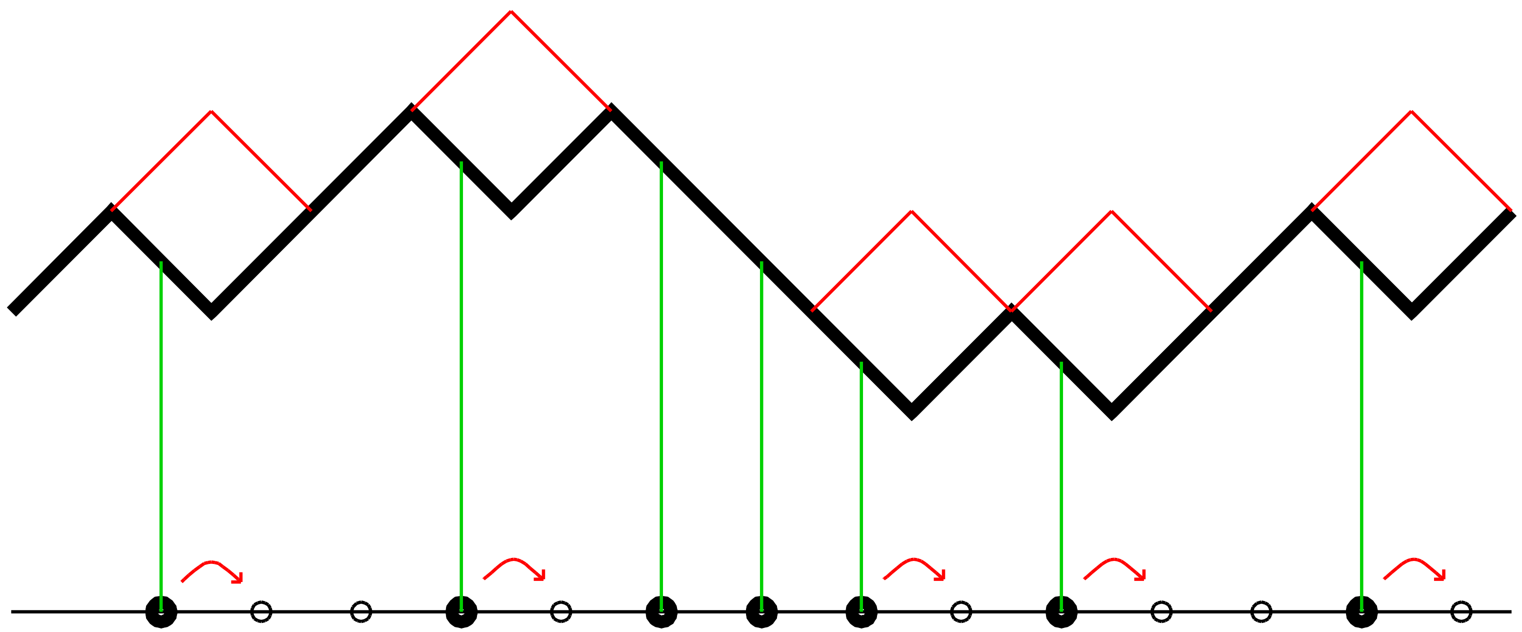
Totally Asymmetric Simple Exclusion Process (TASEP)



Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as $\exp(-x)dx$.

TASEP is a representative of a (conjectural) universality class of growth models in 1+1 dimensions that is characterized by

- Locality of growth (no long-range interaction)
- A smoothing mechanism (a.k.a. relaxation)
- Lateral growth (speed of growth depends on the slope)

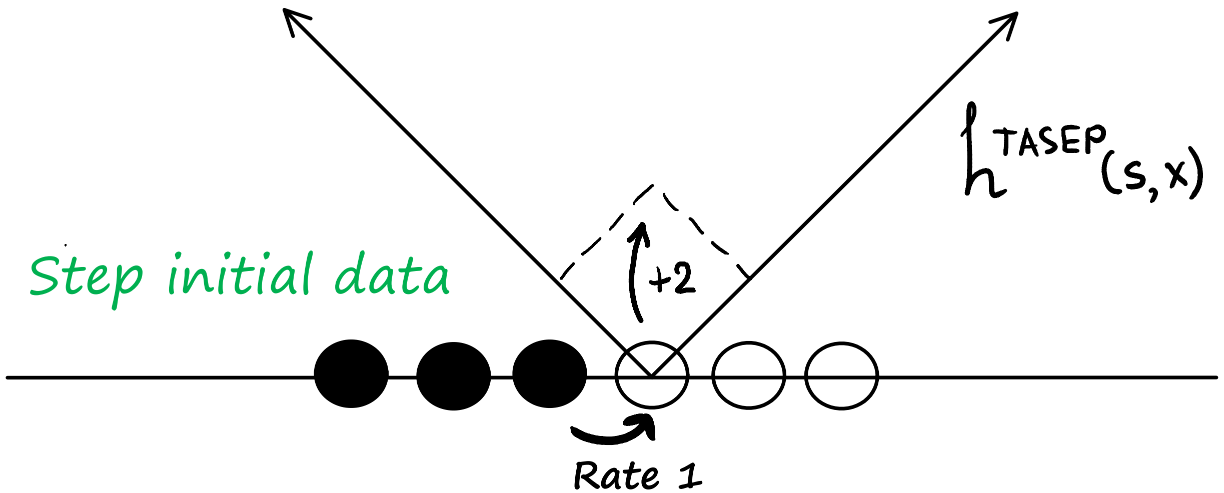
[Kardar-Parisi-Zhang 1986] suggested that growth models with these features should be described by

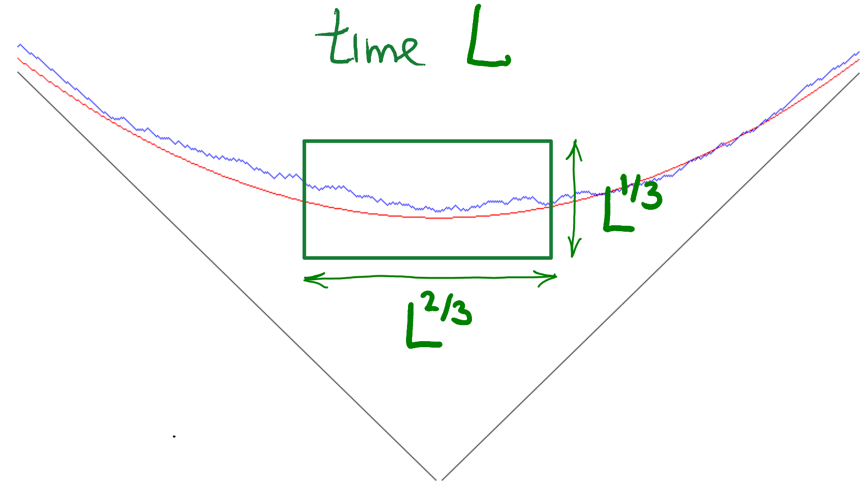
$$\frac{\partial h}{\partial t} = \nu \cdot \frac{\partial^2 h}{\partial x^2} + \lambda \cdot \left(\frac{\partial h}{\partial x}\right)^2 + \sqrt{D} \cdot \dot{W}$$

Here ν, λ, and D are model-specific constants, and $\dot{W}$ is the space-time white noise.

These are the KPZ equation and the KPZ universality class.

TASEP with step initial data





$$h_L(t,x) := \frac{1}{L^{1/3}}\, h^{TASEP}(Lt, L^{2/3}x) - L^{2/3}\cdot\frac{t}{2}$$

Theorem *[Johansson 1999]* *For TASEP with step initial data*

$$\lim_{L\to\infty} \mathbb{P}\{h_L(1,0) \geq -s\} = F_{GUE}(s)$$

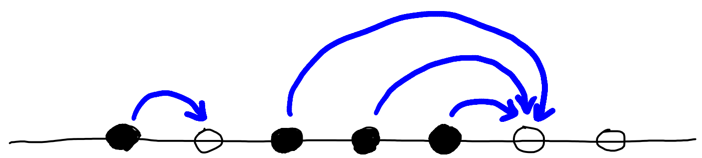
Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian matrices

Limiting (joint) distributions for general t and x are conjecturally universal.

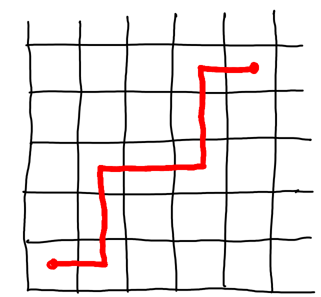
TASEP is one of a few growth models in the KPZ class that can be analyzed via the techniques of determinantal point processes (or free fermions, nonintersecting paths, Schur processes).

Other examples include

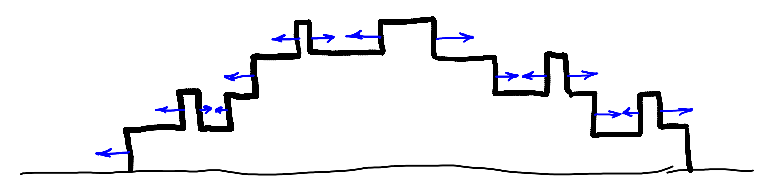
- Discrete time TASEPs with sequential/parallel update
- PushASEP or long range TASEP



- Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights



- Polynuclear growth processes



Recent advances on the KPZ front:

1. Strong experimental evidence that real life systems follow the KPZ class universal laws
2. Direct well-posedness of the KPZ equation and some evidence of universality of the equation
3. Non-determinantal models whose large time behaviour has been analyzed

Non-determinantal models whose limit behaviour has been analyzed:

- ASEP [Tracy-Widom, 2009], [Borodin-Corwin-Sasamoto, 2012]
- KPZ equation / stochastic heat equation (SHE)
  [Amir-C-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+], [Calabrese-Le Doussal-Rosso, 2010+], [B-C-Ferrari, 2012]
- q-TASEP [B-C, 2011+], [B-C-Sasamoto, 2012]
- Semi-discrete stochastic heat equation
  [O'Connell, 2010], [B-C, 2011, B-C-Ferrari, 2012]
- Fully discrete log-Gamma polymer (stochastic heat equation)
  [C-O'Connell-Seppalainen-Zygouras, 2011] [B-C-Remenik, 2012]
- q-PushASEP [B-Petrov, 2013], [C-Petrov, 2013]

Discrete time q-TASEPs

q-pushASEP

q-TASEP

ASEP

log-Gamma discrete polymer

semi-discrete Brownian polymer

KPZ/SHE/continuous Brownian polymer

universal limits (Tracy-Widom distributions, Airy processes)

Macdonald processes $q, t \in [0,1)$

Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t=0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{gl}_N$, $U_q(gl_N)$

Hall–Littlewood processes $q=0$

Random matrices over finite fields

Spherical functions for p-adic groups

General β RMT $t = q^{\beta/2} \to 1$

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Whittaker processes $t=0$, $q \to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$

Kingman partition structures

Cycles of random permutations $q=0$

Poisson–Dirichlet distributions $t=1$

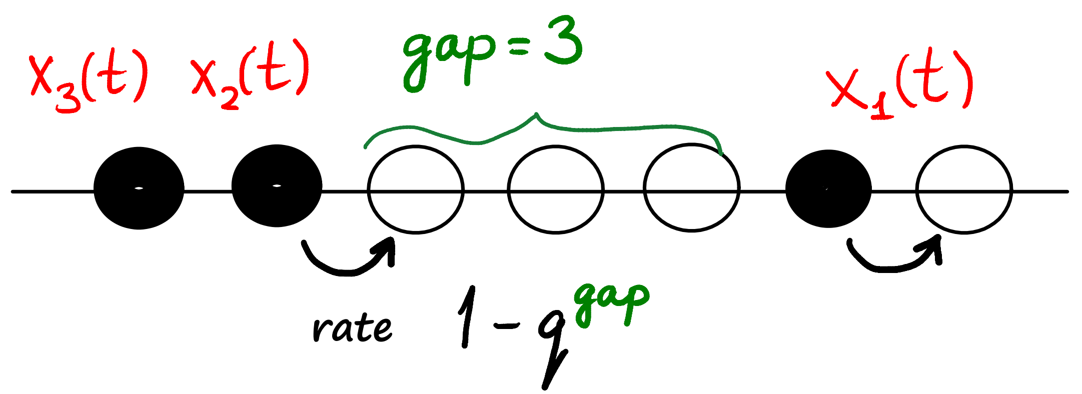
Schur processes $q=t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

Basic reason that all these models turned out to be accessible is the existence of a large family of observables whose averages are explicit.

Example 1: q-TASEP [Borodin-Corwin, 2011]

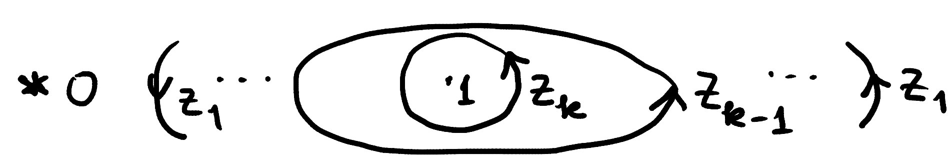


$0 < q < 1$

Theorem [B-C'11], [B-C-Sasamoto'12] *For q-TASEP with step initial data* $\{x_n(0) = -n\}_{n\ge 1}$

$$\mathbb{E}\left[q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\frac{e^{(q-1)tz_j}}{(1-z_j)^{N_j}}\frac{dz_j}{z_j}$$

$(N_1 \ge N_2 \ge \dots \ge N_k)$



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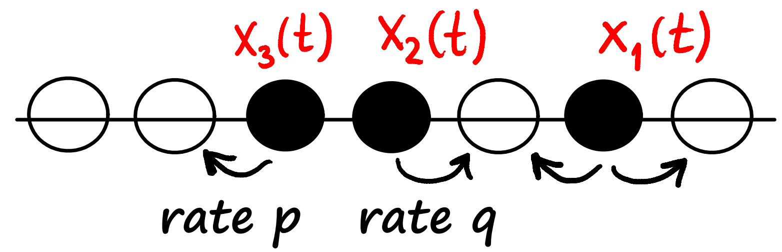
Example 2: ASEP



$0<p<q<1, \quad p+q=1$

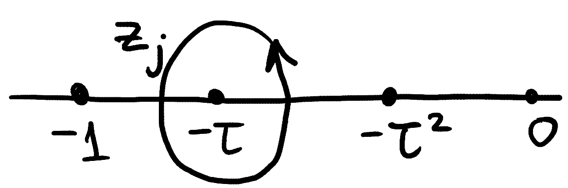
Set $\tau = p/q < 1$, $\quad N_y(t) = \#\{m \ge 1 : x_m(t) \ge y\}$, $\quad Q_y = \dfrac{\tau^{N_y} - \tau^{N_y - 1}}{\tau - 1}$.

Theorem [B-C-Sasamoto, 2012] *For ASEP with step initial data* $\{x_n(0) = -n\}_{n \ge 1}$

$$\mathbb{E}\left[Q_{y_1}(t)\cdots Q_{y_k}(t)\right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - \tau z_B}$$

$(y_1 > y_2 > \cdots > y_k)$

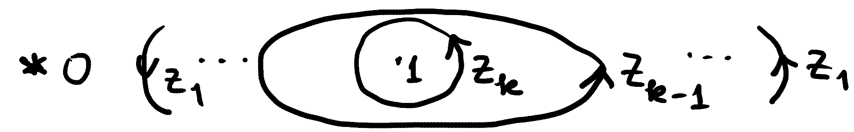
$$* \prod_{j=1}^{k} e^{-\frac{z_j (p-q)^2 t}{(1+z_j)(p+q z_j)}} \left(\frac{1+z_j/\tau}{1+z_j}\right)^{y_j+1} \frac{dz_j}{\tau + z_j}$$



Let us briefly explain why such formulas are useful for asymptotics.

For q-TASEP with step initial data, one specializes to q-moments

$$\mathbb{E}\left(q^{x_N(t)+N}\right)^k = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^N} \frac{dz_j}{z_j}$$



and takes their generating function (q-Laplace transform)

$$\mathbb{E} \frac{1}{\prod_{m\geq 0}\left(1-\zeta q^m q^{x_N+N}\right)} = \mathbb{E} \sum_{k=0}^{\infty} \frac{\left(q^{x_N+N}\right)^k \zeta^k}{(1-q)\cdots(1-q^k)} = \det\left(\mathbb{1}+K\right)_{L^2(\mathbb{N}\times \circlearrowleft)}$$

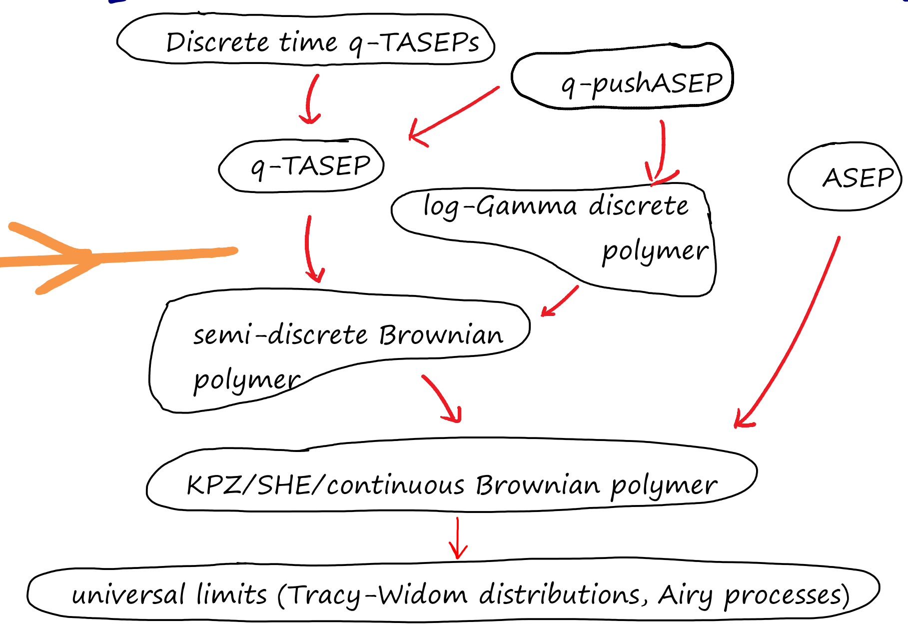
with

$$K(n_1, w_1; n_2, w_2) = \frac{f(w_1)\cdots f(q^{n_1-1} w_1)\,\zeta^{n_1}}{q^{n_1} w_1 - w_2}, \qquad f(w) = \frac{e^{(q-1)\tau w}}{(1-w)^N}$$

The result is suitable for taking various limits.

Example 3: semi-discrete Brownian polymer [O'Connell-Yor, 2001]

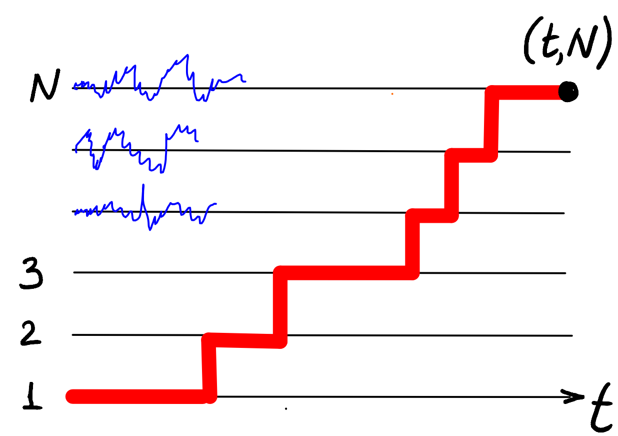
Taking a suitable scaling limit of the q-TASEP as $q \to 1$, one arrives at the following partition functions



$$Z_t^N = \int\limits_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)}\, ds_1 \dots ds_{N-1}$$

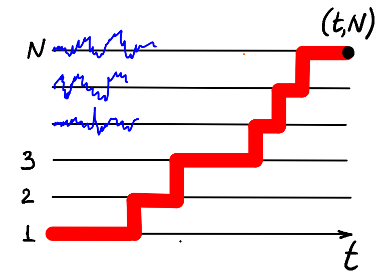
$B_1, \dots, B_N$ are independent BMs

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)\,dx$$



Example 3: semi-discrete Brownian polymer

Theorem [B-C, 2011] The Laplace transform of the polymer partition function Z_t^N *can be written as a Fredholm determinant*



$$\mathbb{E}\left[e^{-uZ_t^N}\right] = \det\left(\mathbb{1} + K_u\right)_{L^2(\odot)}$$

where

$$K_u(v,v') = \frac{i}{2}\int_{-i\infty+\frac{1}{2}}^{i\infty+\frac{1}{2}} \left(\frac{\Gamma(v-1)}{\Gamma(s+v-1)}\right)^N \frac{u^s e^{vts+\frac{ts^2}{2}}}{s+v-v'} \frac{ds}{\sin \pi s}.$$

Corollary [B-C, B-C-Ferrari, 2011-12] *Set* $F_t^N = \log Z_t^N$. *For any* $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F_{\varkappa N}^N - N f_\varkappa}{N^{1/3}} \le r \right\} = F_{GUE}\left(\left(\frac{g_\varkappa}{2}\right)^{-1/3} r \right)$$

Here one cannot use the moment expansion of the Laplace transform because it is divergent! Leads to the (non-rigorous) replica trick.

Currently two approaches to observables with explicit averages:

1. Quantum many body systems
2. Macdonald processes

While first seems more universal; second is much more conceptual.

In the case of KPZ/SHE they go back to two basic developments:

1. In physics: Bethe ansatz solution of the quantum delta-Bose gas [Lieb-Liniger, 1963], [Bethe, 1931]
2. In math: Plancherel theorem for semisimple Lie groups [Harish-Chandra, 1958]

We will now focus on Macdonald processes.

Macdonald processes $q, t \in [0,1)$

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Characters of symmetric, unitary groups

(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand-Tsetlin patterns)

$$
\begin{array}{ccccccccc}
\lambda^{(N)}_{N} & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_{2} & & \lambda^{(N)}_{1} \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & & \cdots & & \lambda^{(N-1)}_{1} & \\
& & \ddots & & \cdots & & ⋰ & & \\
& & & \lambda^{(2)}_{2} & & \lambda^{(2)}_{1} & & & \\
& & & & \lambda^{(1)}_{1} & & & &
\end{array}
$$

$$\lambda^{(m)}_{j} \in \mathbb{Z}_{\geq 0}$$

Macdonald polynomials

$$\mathbb{P}\left(\lambda^{(k)}\right) = \frac{P_{\lambda^{(k)}}(a_1, \dots, a_k)\, Q_{\lambda^{(k)}}(b_1, \dots, b_M)}{\Pi(\underbrace{a_1, \dots, a_k}; \underbrace{b_1, \dots, b_M})}$$

normalization constant

two groups of parameters

Macdonald polynomials $P_\lambda(x_1,\dots,x_N)\in\mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ with partitions $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$(\mathcal{D}_1 f)(x_1,\dots,x_N)=\sum_{i=1}^{n}\prod_{j\neq i}\frac{t x_i-x_j}{x_i-x_j}\,f(x_1,\dots,qx_i,\dots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda=(q^{\lambda_1}t^{N-1}+q^{\lambda_2}t^{N-2}+\dots+q^{\lambda_N})P_\lambda .$$

They have many remarkable properties that include orthogonality (dual basis Q_λ), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with $\mathcal{D}_1$, etc.

We are able to do two basic things:

- Construct relatively explicit Markov operators that map Macdonald processes to Macdonald processes;
- Evaluate averages of a broad class of observables.

The construction is based on commutativity of Markov operators

$$\mathbb{P}(\lambda \to \mu) = \frac{P_\mu(x_1, \dots, x_{n-1})}{P_\lambda(x_1, \dots, x_n)} P_{\lambda/\mu}(x_n), \qquad \mathbb{P}(\lambda \to \nu) = \frac{P_\nu(x_1, \dots, x_m)}{P_\lambda(x_1, \dots, x_m)} \frac{P_{\nu/\lambda}(u)}{\Pi(x; u)}$$

skew Macdonald polynomials

additional parameter

an idea from [Diaconis-Fill '90], and Schur process dynamics from [Borodin-Ferrari '08].

Evaluation of averages is based on the following observation.
Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

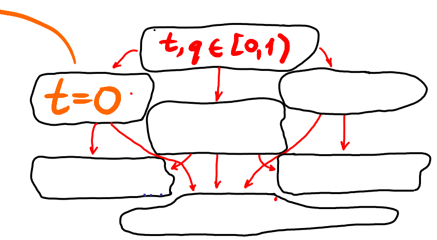
$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$
we obtain

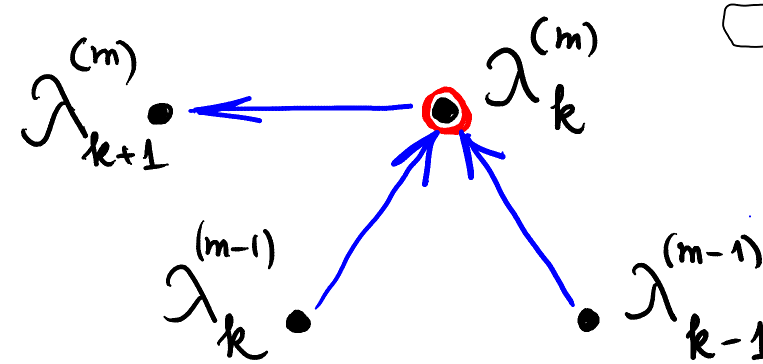
$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.

Here is an example of a Markov process preserving the class of the q-Whittaker processes (Macdonald processes with $t=0$).

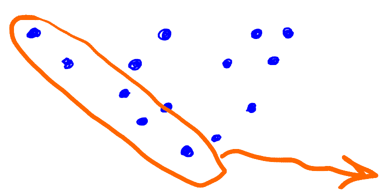


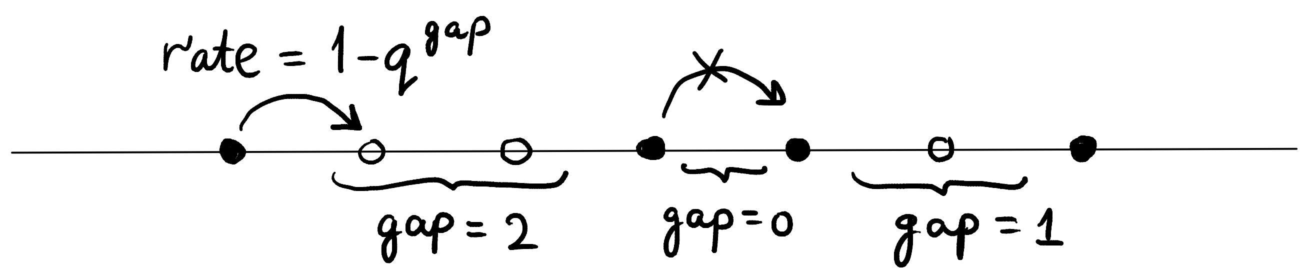
Each coordinate of the triangular array jumps by 1 to the right independently of the others with



$$\mathrm{rate}\left(\lambda^{(m)}_k\right) = \frac{\left(1-q^{\lambda^{(m-1)}_{k-1}-\lambda^{(m)}_k}\right)\left(1-q^{\lambda^{(m)}_k-\lambda^{(m)}_{k+1}+1}\right)}{\left(1-q^{\lambda^{(m)}_k-\lambda^{(m-1)}_k}\right)}.$$

The set of coordinates $\left\{\lambda^{(m)}_m - m\right\}_{m\geq 1}$ forms q-TASEP

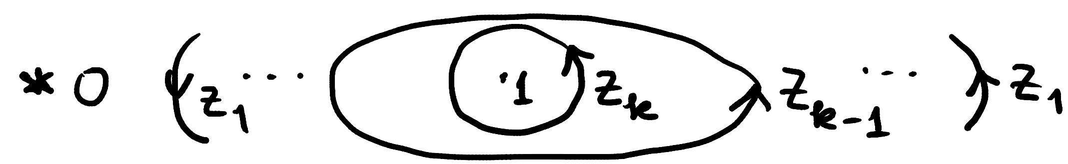




Taking the observables corresponding to products of the first order Macdonald operators on different levels results in the integral representation for the q-moments of the q-TASEP particle positions (the eigenvalues are $q^{x_{N_j}+N_j}$)

$$\mathbb{E}\left[q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)}\right] = \frac{(-1)^k\, q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^{N_j}}\frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \dots \geq N_k)$



Computing the residues at $z_k=1$; $z_{k-1}=1,q$; $z_{k-2}=1,q,q^2$; etc. provides a direct link to the difference operators.

Let us briefly look at the quantum many body problems approach.

It is not hard to check that for the q-TASEP

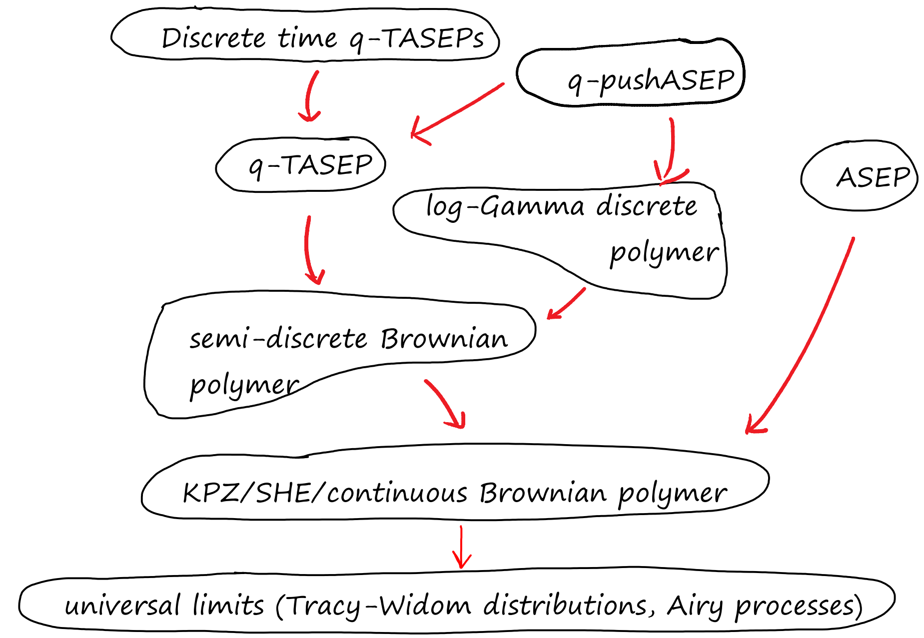
$$V(t;N_1,\dots,N_k)=\mathbb{E}\left[q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)}\right]$$

satisfies [B-C-Sasamoto '12]

$$\partial_t V = (1-q)\left(\sum_{i=1}^{k}\nabla_i + (1-q^{-1})\sum_{i<j}\mathbb{1}_{N_i=N_j}\,q^{j-i}\,\nabla_i\right)V$$

where $(\nabla_i f)(N_1,\dots,N_k)=f(N_i-1)-f(N_i)$.

One easily verifies that the nested contour integrals satisfy this (closed) system of equations with desired initial data, thus proving the formulas in an elementary way.



Taking the limit to the continuous Brownian polymer, one obtains that if $Z(t,x)$ solves the Stochastic Heat Equation

space-time white noise

$$\partial_t Z = \frac{1}{2}\partial_{xx}^2 Z + \dot{W} Z, \qquad Z(0,x) = \delta(x),$$

then $\quad \bar{Z}(t;x_1,\dots,x_k) = \mathbb{E}\left[Z(t,x_1)\cdots Z(t,x_k)\right] \quad$ satisfies the

quantum delta-Bose gas evolution [Molchanov '86], [Kardar '87]

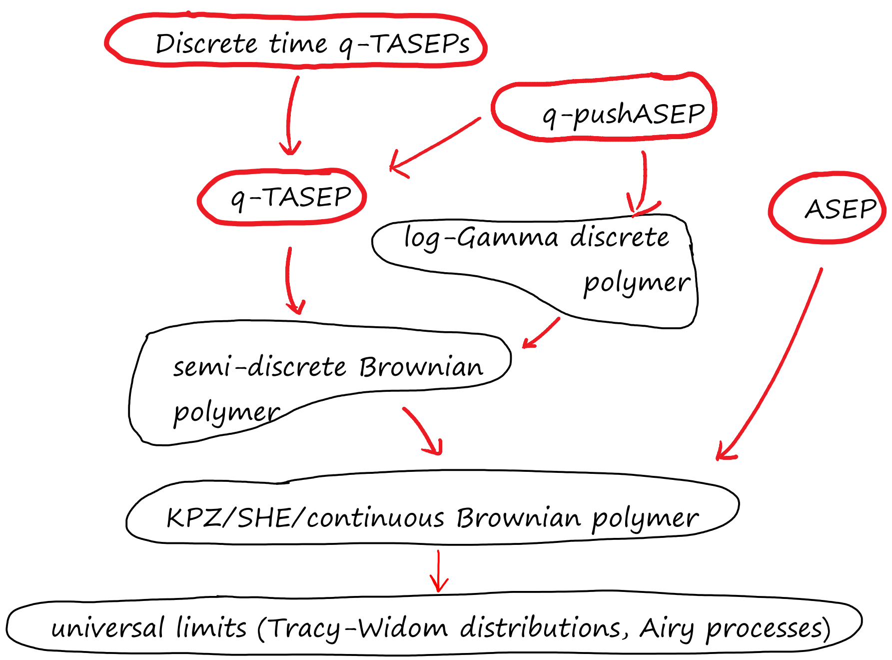
$$\partial_t \bar{Z} = \frac{1}{2}\Big(\Delta + \sum_{i\neq j}\delta(x_i - x_j)\Big)\bar{Z}$$

The quantum delta-Bose gas evolution equation

$$\partial_t \bar{Z} = \frac{1}{2}\Big(\Delta + c\sum_{i\neq j}\delta(x_i - x_j)\Big)\bar{Z}$$

is solvable via Bethe ansatz [c<0 Lieb-Liniger '63, c>0 McGuire '64].
Gives moments of Z, and, via the (non-rigorous) replica trick, the Laplace transform of Z [Dotsenko '10], [Calabrese et.al. '10].

This can be seen as a limiting case of a rigorous argument at the level of the q-TASEP and ASEP [B-C-Sasamoto '12].



To summarize:

- There is a new set of rigorously analyzable growth models in (1+1)d in the KPZ universality class. They are characterized by having large families of observables with explicit averages.
- One way of obtaining such models is via marginals of the Macdonald processes that manufacture the random growth and the observables from the structural properties of the Macdonald polynomials.
- Another approach is to notice that the averages solve quantum many body systems that are integrable. However, this provides no way of knowing which growth models and/or which observables would fit.
- Turning the averages into asymptotic information remains challenging
- More cases of Macdonald processes are to be investigated