The renormalization group according to Balaban

J. Dimock
SUNY at Buffalo

May 27, 2013

Outline:

- Overview
- Results by Balaban
- Exposition of the method for ϕ_3^4 (some new features)
- Technical remarks

Overview

- $\mathbb{T}_M^{-N} = \Big(L^{-N}\mathbb{Z}/L^M\mathbb{Z} \Big)^d$
  $=$ toroidal lattice with spacing L^{-N}, linear dimension L^M.
- fields $\varphi : \mathbb{T}_M^{-N} \to \mathbb{R}$
- Euclidean action $\mathcal{A}(\varphi)$
- partition function $Z_{M,N} = \int \exp\Big(-\mathcal{A}(\varphi)\Big) \prod_x d\varphi(x)$
- correlation functions
  $Z_{M,N}^{-1} \int \varphi(x_1)\dots\varphi(x_n) \exp\Big(-\mathcal{A}(\varphi)\Big) \prod_x d\phi(x)$
- UV problem - bounds uniform in N.
  IR problem - bounds uniform in M

Balaban projects

1. Scalar electrodynamics in d=2,3 (UV problem, 5 papers, 1982-83)
2. Yang-Mills in d=3,4 (UV problem, 11 papers, 1984-89)
3. Linear σ-model in $d \geq 3$ (IR problem, 8 papers, 1995-99)

(Also with J. Imbrie, A. Jaffe: partial results on abelian Higgs model in d=2,3 (UV and IR, 1988))

1. Scalar electrodynamics in d=2,3

- $\varphi : \mathbb{T}_M^{-N} \to \mathbb{C}$ $\qquad$ $A :$ bonds in $\mathbb{T}_M^{-N} \to \mathbb{R}$
- with $\int_x \cdots = L^{-Nd} \sum_x \ldots$

$$\mathcal{A}(\varphi, A) = \frac{1}{2} \int \Big(|\partial_A \varphi|^2 + m^2 |\varphi|^2 + \lambda |\phi|^4 + |\partial A|^2 + \mu^2 |A|^2 \Big)$$

Note: massive photons

- Results (after renormalization):
  - bounds on effective densities for RG flow uniform in N
  - stability bound:

$$\exp\Big(- c\mathrm{Vol}(\mathbb{T}_M^{-N} \Big) \leq \frac{Z_{M,N}}{Z_{M,N}(0)} \leq \exp\Big(c\mathrm{Vol}(\mathbb{T}_M^{-N}) \Big)$$

with constants independent of N, M.

2. Yang-Mills in d=3,4

- U: bonds in $\mathbb{T}_M^{-N} \to$ semi-simple compact Lie group G
- $\mathcal{A}(U) = \frac{1}{g^2}\int_p tr\Big(1 - U(\partial p)\Big) \qquad U(\partial p) = \prod_{b \in \partial p} U(b)$
- Results (after renormalization):
  - bounds on effective densities for RG flow
  - upper stability bound
- Caveat: paper on coupling constant flow in second order perturbation theory missing

3. Linear σ-model in $d \geq 3$

- $\varphi : \mathbb{T}^0_M \to \mathbb{R}^n \qquad n \geq 2$
- $\mathcal{A}(\phi) = \beta\Big(|\partial\varphi|^2 + \lambda(|\varphi|^2 - 1)^2\Big)$
- Results : For β large, $1 \leq \lambda \leq \infty$ and uniformly in M:
    1. bounds on effective densities for RG flow
    2. spontanenous magnetization
    3. massless decay of correlations (Goldstone bosons)

(Last part with M. O'Carroll)

φ_3^4 model - the UV problem

- $\mathbb{T}_M^{-N} = \left(L^{-N}\mathbb{Z}/L^M\mathbb{Z}\right)^3$ $\qquad \varphi : \mathbb{T}_M^{-N} :\rightarrow \mathbb{R}$

$$\mathcal{A}(\varphi) = S(\varphi) + V(\varphi)$$

$$S(\varphi) = \frac{1}{2}\|\partial\varphi\|^2 + \frac{1}{2}\bar{\mu}\ \|\varphi\|^2$$

$$V(\varphi) = \varepsilon \mathrm{Vol}(\mathbb{T}_M^{-N}) + \frac{1}{2}\mu\ \|\varphi\|^2 + \frac{1}{4}\lambda \int \varphi^4$$

- Scale to $\Phi : \mathbb{T}^0_{M+N} :\rightarrow \mathbb{R}$ (Now IR problem)

$$\mathcal{A}_0(\Phi) = S_0(\Phi) + V_0(\Phi)$$

$$S_0(\Phi) = \frac{1}{2}\|\partial\Phi\|^2 + \frac{1}{2}\bar{\mu}_0\ \|\Phi\|^2$$

$$V(\Phi) = \varepsilon_0 \mathrm{Vol}(\mathbb{T}^0_{M+N}) + \frac{1}{2}\mu_0\ \|\Phi\|^2 + \frac{1}{4}\lambda_0 \int \Phi^4$$

with tiny coupling constants:

$$\varepsilon_0 = L^{-3N}\varepsilon \qquad \mu_0 = L^{-2N}\mu \qquad \lambda_0 = L^{-N}\lambda$$

RG transformation: define densities $\rho_k(\Phi_k)$ on $\Phi_k : \mathbb{T}^0_{M+N-k} \to \mathbb{R}$ by $\rho_0 = \exp(-\mathcal{A}_0(\Phi))$ and

$$\rho_{k+1}(\Phi_{k+1}) = \mathcal{N}_{k+1}^{-1} \int \exp\Big(- \frac{a}{L^2} \|\Phi_{k+1,L} - Q\Phi_k\|^2 \Big) \rho_k(\Phi_k) d\Phi_k$$

where

- For $y \in \mathbb{T}^1_{M+N-k}$

$$(Q\Phi_k)(y) = L^{-3} \sum_{x:|x-y|<L/2} \Phi_k(x)$$

- $\Phi_{k+1,L}(x) = L^{-1/2}\Phi_{k+1}(x/L)$
- $\mathcal{N}_{k+1}$ is chosen so

$$\int \rho_{k+1}(\Phi_{k+1}) d\Phi_{k+1} = \int \rho_k(\Phi_k) d\Phi_k$$

Can calculate partition function from any ρ_k.

First step:

- $$\rho_1(\Phi_1) = \mathcal{N}_1^{-1} \int \exp\Big(-\frac{a}{L^2}\|\Phi_{1,L} - Q\Phi_0\|^2 - S_0(\Phi_0) - V_0(\Phi_0)\Big) d\Phi_0$$

- First two terms in exponential have minimum at

$$\Phi_0 = a\Big(-\Delta + \bar{\mu}_0 + \frac{a}{L^2}Q^TQ\Big)^{-1} Q^T\Phi_{1,L} \equiv \varphi_{1,L}$$

where $\varphi_1 : \mathbb{T}^{-1}_{M+N-1} \to \mathbb{R}$. ($\varphi_1$ is Φ_1 smeared on finer lattice.)

- Expand around minimizer $\Phi_0 = \phi_{1,L} + Z$. Integrate over Z.

- $$Z_1 \exp(-S_1(\Phi_1)) \int \exp\Big(-V_0(\phi_{1,L}+Z)\Big) d\mu_{C_1}(Z)$$

  where measure is Gaussian with covariance

  $$C_1 = \Big(-\Delta + \bar{\mu}_0 + \frac{a}{L^2} Q^T Q\Big)^{-1}$$

- cluster expansion, perturbation theory?
- Problem: need small coupling constants, small fields

Remedy: Insert under integral sign:

$$1 = \sum_{\Omega} \zeta(\Omega^c, \Phi_0, \Phi_1)\chi(\Omega, \Phi_0, \Phi_1)$$

where Ω is union of cubes (not necessarily connected) and

- $\chi(\Omega, \Phi_0, \Phi_1)$ enforces that in every cube in Ω we have

$$|\Phi_{1,L} - Q\Phi_0| \leq |log \lambda_0|^p$$

$$|\partial\Phi_0| \leq |log \lambda_0|^p$$

$$|\Phi_0| \leq \lambda_0^{-1/4}|log \lambda_0|^p$$

- $\zeta(\Omega^c, \Phi_0, \Phi_1)$ enforces that in every cube in Ω^c there is at least one point where one of these inequalities is violated.

$$\rho_1(\Phi_1) = \sum_{\Omega}$$

$$\int d\Phi_{0,\Omega^c}\, \zeta(\Omega^c) \exp\left(-\frac{a}{L^2}\|\Phi_{1,L} - Q\Phi_0\|^2_{\Omega^c} - S_0(\Omega^c, \Phi_0) - V_0(\Omega^c, \Phi_0)\right)$$

$$\int d\Phi_{0,\Omega}\, \chi(\Omega) \exp\left(-\frac{a}{L^2}\|\Phi_{1,L} - Q\Phi_0\|^2_{\Omega} - S_0(\Omega, \Phi_0) - V_0(\Omega, \Phi_0)\right)$$

Large field integral :

- Get factor $\exp(-|log \lambda_0|^p) = \mathcal{O}(\lambda_0^n)$ for every cube in Ω^c.
- Hence overall $\exp(-|log \lambda_0|^p |\Omega^c|)$.
- Sufficient to control sum over Ω.

Small field integral: block averaging supplies a mass for cluster expansion. Get

$$\mathcal{Z}_\Omega \exp\Big(- S_1(\Omega, \Phi_1) - V_1(\Omega, \varphi_{1,\Omega}) + \sum_{X \subset \Omega} E_1(X, \varphi_{1,\Omega})\Big)$$

$$V_1(\Omega, \varphi) = \varepsilon_1 \mathrm{Vol}(\Omega) + \frac{1}{2}\mu_1 \|\phi\|^2_\Omega + \frac{1}{4}\lambda_1 \int_\Omega \phi^4$$

where

1. X is connected union of cubes = "polymer"
2. $E_1(X, \varphi)$ depends on φ only in X.
3. $E_1(X, \varphi) = \mathcal{O}(e^{-\kappa|X|})$.
4. $E_1(X, \varphi)$ has local parts removed.

Repeat first step. After k steps have lattice $\mathbb{T}^{-k}_{M+N-k}$ with larger coupling constant

$$\lambda_k = L^k \lambda_0 = L^{-(N-k)}\lambda$$

At each step new small field region Ω_k defined by λ_k.

$$\boldsymbol{\Omega} = (\Omega_1, \cdots \Omega_k) \qquad \Omega_1 \supset \Omega_2 \supset \cdots \Omega_k \qquad \Omega_j = L^{-(k-j)} \text{ cubes}$$

active fields

$$\Phi_{k,\boldsymbol{\Omega}} = (\Phi_{0,\Omega_1^c}, \Phi_{1,\delta\Omega_1}, \dots, \Phi_{k-1,\delta\Omega_{k-1}}, \Phi_{k,\Omega_k}) \qquad \delta\Omega_j = \Omega_j - \Omega_{j+1}$$

smeared fields: $(Q_{k,\boldsymbol{\Omega}} = (Q_{\delta\Omega_1}, Q^2_{\delta\Omega_2}, \dots, Q^k_{\Omega_k}))$

$$\varphi_{k,\boldsymbol{\Omega}} = a_k \Big[-\Delta + \bar{\mu}_k + Q^T_{\boldsymbol{\Omega}} \mathbf{a} Q_{\boldsymbol{\Omega}} \Big]^{-1}_{\Omega_1} Q^T_{\boldsymbol{\Omega}} \Phi_{k,\boldsymbol{\Omega}} + \dots$$

and interaction

$$V_k(\Omega_k, \varphi_{k,\boldsymbol{\Omega}}) = \varepsilon_k \mathrm{Vol}(\Omega_k) + \frac{1}{2}\mu_k \|\varphi_{k,\boldsymbol{\Omega}}\|^2_{\Omega_k} + \frac{1}{4}\lambda_k \int_{\Omega_k} \varphi^4_{k,\boldsymbol{\Omega}}$$

After k steps

$$\rho_k(\Phi_k) = \sum_{\mathbf{\Omega}=(\Omega_1,\cdots\Omega_k)} \prod_{j=0}^{k-1} \int d\Phi_{j,\Omega_{j+1}^c} \exp\Big(-a_j \|\Phi_{j+1,L} - Q\Phi_j\|^2_{\Omega_{j+1}^c}\Big)$$

$$\prod_{j=0}^{k-1} \exp\Big(-\tilde{S}_j(\delta\Omega_j) - \tilde{V}_j(\delta\Omega_j) + \sum_{X\subset\Omega_{j+1}^c, X\cap\delta\Omega_j\neq\emptyset} \tilde{B}_{j,\mathbf{\Omega}_j}(X)\Big)$$

$$\mathcal{Z}_{\mathbf{\Omega}} \exp\Big(-S_k(\Omega_k) - V_k(\Omega_k) + \sum_{X\subset\Omega_k} E_k(X) + \sum_{X\#\Omega_k} B_{k,\mathbf{\Omega}}(X)\Big)$$

$$\times \text{ various characteristic functions}$$

In $V_k(\Omega_k,\phi) = \varepsilon_k \mathrm{Vol}(\Omega_k) + \frac{1}{2}\mu_k \|\phi\|^2_{\Omega_k} + \frac{1}{4}\lambda_k \int_{\Omega_k} \phi^4$ find RG flow:

$$\varepsilon_{k+1} = L^3 \varepsilon_k + \varepsilon^*_k(\lambda_k, \mu_k, E_k)$$

$$\mu_{k+1} = L^2 \mu_k + \mu^*_k(\lambda_k, \mu_k, E_k)$$

$$\lambda_{k+1} = L\ \lambda_k$$

$$E_{k+1} = \mathcal{L}(E_k) + E^*_k(\lambda_k, \mu_k, E_k)$$

$\|\mathcal{L}\| < 1$ if we use norm

$$\|E_k\|_k = \sup_X \left[\sup_{|\phi|,|\partial\phi| \leq \lambda_k^{-\frac{1}{4}}} |E_k(X,\phi)| \right] e^{\kappa|X|}$$

Theorem: (Non-perturbative renormalization) Let λ be sufficiently small. Then there exists a unique (bounded) solution for $0 \leq k \leq N$ with $\varepsilon_N = 0, \mu_N = 0$ and $E_0 = 0$.

proof: Existence equivalent to existence of a fixed point of certain operator on space of sequences $\{\mu_k, E_k\}$.

Show operator is contraction.

Fixed point theorem.

After N steps we return to $\mathbb{T}_M^{-N}$ with $\lambda_N = \lambda$.
Last integral needs no block averaging

$$Z_{M,N} = \int \rho_N(\Phi_N)\ d\Phi_N$$

Final cluster expansion. Combine all terms that yield a fixed Ω_N.

$$Z_{M,N} = Z_{M,N}(0) \sum_{\Omega_N} \rho'(\Omega_N^c) \exp\Big(\sum_{X \cap \Omega_N \neq \emptyset} E'_N(X) \Big)$$

$\rho'(\Omega_N^c)$ factors over connected components. After resummation

$$Z_{M,N} = Z_{M,N}(0) \exp\Big(\sum_{X \subset \mathbb{T}_M^{-N}} E''_N(X) \Big)$$

with $\|E''_N\|_N = \mathcal{O}(\lambda^{1/4})$

Theorem (stability bound): For renormalized φ_3^4:

$$\exp\Big(-c\mathrm{Vol}(\mathbb{T}_M^{-N})\Big) \leq \frac{Z_{M,N}}{Z_{M,N}(0)} \leq \exp\Big(c\mathrm{Vol}(\mathbb{T}_M^{-N})\Big)$$

with constants independent of N, M.

History

- Glimm, Jaffe (1973) - upper bound
  Feldman-Osterwalder (1975) - $M \to \infty$ limit
- Balaban (1983) - upper and lower bound
- Brydges, Dimock, Hurd (1995) - new proof upper
- Present work (2011)

How much perturbation theory?

- Glimm, Jaffe (1973) ....High order
- Balaban (1983) ....Medium order
- Brydges, Dimock, Hurd (1995) ....Second order
- Present work (2011) ....Zeroth order

Remarks:

- Actual proof more complicated
- Method should work for correlation functions, $N \to \infty$ limit, $M \to \infty$ limit.
- Method should work for QED_3 as well

References:

1. The renormalization group according to Balaban.
   I. Small fields. ArXiv 1108.1335.
2. The renormalization group according to Balaban.
   II. Large fields. ArXiv 1212.5562
3. The renormalization group according to Balaban.
   III. Convergence . ArXiv 1304.0705

Technical remarks

Locality and smoothness of fields?

Fields after k steps field on $\mathbb{T}^{-k}_{M+N-k}$

$$\varphi_{k,\mathbf{\Omega}} = G_{k,\mathbf{\Omega}} Q^T_{\mathbf{\Omega}} a_{\mathbf{\Omega}} \Phi_{k,\mathbf{\Omega}} \; + \; G_{k,\mathbf{\Omega}} \Delta_{\Omega_1,\Omega_1^c} \Phi_{0,\Omega_1}$$

where

$$G_{k,\mathbf{\Omega}} = \Big[-\Delta + \bar{\mu}_k + \sum_{j=1}^{k} [Q_j^T a_j L^{2(k-j)} Q_j]_{\delta\Omega_j} \Big]^{-1}_{\Omega_1}$$

Effective mass $L^{(k-j)}$ in $\delta\Omega_j$ (soft boundary conditions).

$G_{k,\mathbf{\Omega}}(x,y)$ has exponential decay, bounds on derivatives order < 2.

Strictly localized fields?

Systematically replace $\mathbf{\Omega}$ by

$$\mathbf{\Omega}' = \text{ minimal } \mathbf{\Omega} \text{ around final } \Omega_k$$

$$\phi_{k,\mathbf{\Omega}'} - \phi_{k,\mathbf{\Omega}} = \mathcal{O}(\lambda_k^n) \qquad \text{in } \Omega_k$$

Then $\phi_{k,\mathbf{\Omega}'}$ is strictly localized near Ω_k, still smooth

Localize within Ω_k?
Random walk expansion:

$$G_{k,\mathbf{\Omega}}(x,y) = \sum_{\omega: x \to y} G_{k,\mathbf{\Omega},\omega}(x,y)$$

where

$$\omega = (\Box_0, \Box_1, \dots \Box_n)$$

is a sequence of (multiscale, overlapping) blocks and

$$G_{k,\mathbf{\Omega},\omega} = \prod_{j=0}^{n} K_{k,\mathbf{\Omega},\Box_j}$$

Use in cluster expansion for for fields and fluctuation covariance